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A Public Announcement Separation Logic

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We define a Public Announcement Separation Logic, denoted PASL, that allows us to consider epistemic possible worlds as resources that can be shared or separated, in the spirit of separation logics. After studying its semantics and illustrating its interest for modelling systems, we provide a sound and complete tableau calculus that deals with resource, agent and announcement constraints and give also a countermodel extraction method.

1. Introduction

Epistemic Logic is the logic of *knowledge* and *belief*, which models and expresses properties of knowledge that multiple agents may have about themselves and about each other [Hintikka 1962,Moses et al. 1995,Lenzen 1978,Meyer and van der Hoek 1995,van Ditmarsch et al. 2015]. The models of epistemic logic are based on *possible worlds*, possible worlds, that encode the possible states/configurations of a considered system. For instance, in the case of the muddy children problem [Moses et al. 1995], a possible world corresponds to a distribution of mud over foreheads of children; in the case of card games, a possible world corresponds to a deal of cards [van Ditmarsch et al. 2003]; in logics of propositional control, a possible world is a collection of propositional variables that can be controlled by different agents [van der Hoek et al. 2005]. Cards, children, control variables can alternatively be considered as *resources*, and as such are entities that can be composed or decomposed into sub-entities. We can combine individual cards into a deal of cards over agents. We can model the behaviour of individual children observing other children (that may or may not be muddy), and from their individual behaviour emerges (sub)group behaviour. Individual agents controlling subsets of variables can form coalitions that subsequently can control larger sets of variables. Our aim here consists in enriching the models of epistemic logic with more structure namely by considering the propositions that are evaluated in possible worlds as resources that be combined or separated. We want also to investigate the kind of properties that we are then able to express.

In order to model and express properties on resources, various resource logics have been proposed, such as Linear Logic (LL) [Girard 1987] that focuses on resource consumption, and more

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recently the logic of Bunched Implications (BI) and its variants, like Boolean BI (BBI) [Pym 2002], that mainly focus on resource sharing and separation. BI logic combines intuitionistic propositional additives ($\wedge$ and $\to$) with intuitionistic propositional multiplicatives ($*$ and $-\!*$). Let us note that in BBI the additives are classical. A key feature of BI as modelling tool, and hence of its specific model, called Separation Logic (SL) [Ishtiaq and O'Hearn 2001], is its control of the representation and handling of resources provided by the resource semantics and the associated proof systems. For instance, the additive conjunction, $\wedge$, is defined by $r \models \varphi \wedge \psi$ iff $r \models \varphi$ and $r \models \psi$. The key point here is that the resource r is *shared* between the two components of the disjunction. In contrast, the multiplicative conjunction, $*$, divides the resource between its propositional components, using a partial commutative monoidal operation, denoted $\bullet$: $r \models \varphi * \psi$ iff there exist r_1 and r_2 such that $r = r_1 \bullet r_2$ and $r_1 \models \varphi$ and $r_2 \models \psi$. That is, the monoid specifies a *separation* of the resources between the components of the conjunction. In SL, where the semantics is built out of sets of memory locations, the two resource components are required to be disjoint. BI and BBI are the logical kernels of various separation logics, for instance with resources being memory areas in SL [Ishtiaq and O'Hearn 2001,Reynolds 2002], or with resources being located on trees [Biri and Galmiche 2003], or of bunched modal logics modelling dynamic systems that manipulate resources [Collinson and Pym 2009,Courtault and Galmiche 2013].
In this context, as possible worlds are implicitly related to resources, it seems natural to try to extend some epistemic logics with separation connectives of BBI. A first attempt has been proposed in [Courtault et al. 2015] with an Epistemic Separation Logic (ESL), that is a conservative extension of Epistemic Logic and also of BBI in which possible worlds are seen as resources.

In this paper we define a new epistemic separation logic, called Public Annoucement Separation Logic (PASL), that extends this logic with public announcements. After an analysis in order to fix the semantics of the logic and its right definition we first show that ESL is equally expressive as PASL. Then we develop an example emphasizing the power of PASL for modelling and complete this study about expresivity with the proposal of new modalities combining epistemic and separation connectives. The addition of dynamic modalities to a logical language often results in a so-called larger *update expressivity* (what kind of model transformations are possible with the dynamic modalities), even when the formal expressivity of the language (what kind of properties of models can be expressed in the logical language) remains unchanged. This is indeed the case for our logic PASL with respect to the prior proposal ESL. The public announcements in PASL are model transformers, as usual [van Ditmarsch et al. 2007], although on resource monoids special care is needed in the semantics of public announcements to guarantee preservation of the monoid character (as we will see). As ESL does not contain dynamic modalities, obviously the update expressivity of PASL is larger than that of ESL. On the other hand, the logics are equally expressive in the usual sense, as any formula with dynamic announcement modalities is equivalent to a formula without (by an axiomatic rewrite procedure, see Proposition 2.5, later). The advantage of PASL is then the availability of more succinct descriptions of systems behaviour; more precisely, a PASL specification that after some announcements a certain property holds is equivalent to some ESL formula, i.e., without announcements, but at the risk of the latter being exponentially longer than the former. We have not investigated this for PASL, but it is known that the addition of dynamics to similar logics tends to make them more succinct [Lutz 2006,French et al 2013].

In order to complete this study we define a tableaux calculus for the logic, in the spirit of the tableaux calculi for BI and BBI [Galmiche et al. 2005,Larchey-Wendling 2016], with a focus on the countermodel extraction. It is based on specific labels and resource, agent and announcement constraints and adequate closure conditions introduced for this logic. The soundness and completeness properties are proved with concepts and techniques adapted from our works on modal extensions of BI and BBI logics [Courtault and Galmiche 2015] and a countermodel extraction method is also provided. Viewed from a different perspective, our proposal is also an attempt to enrich some separation logics, here BBI, with uncertainty over composition and decomposition of resources, and by different agents. Here we consider separation models through BBI logic and its general semantics but we expect to study such an extension with other resource logics and models with separation, like SL based on memory models and dedicated to program verification. Let us note that we consider BBI logic in which the conjunction is distributive over the disjunction, property that does not hold in LL. Concerning the links between Epistemic Logic and resource management we can mention some works based on Linear Logic that attempt to capture agent knowledge evolutions due to *epistemic actions* [Marion and Sadrzadeh 2003,Baltag et al. 2006], but these works consider the epistemic actions as resources (but not the worlds). Compared with such works, our epistemic separation logic considers the possible (epistemic) worlds as resources, including sharing and separation connectives that allow us to express properties, like for instance $(A \wedge (B \vee C)) \mathrel{-\!\!*} K_a D$ that means that "the addition of a resource that satisfies the property A and also the property B or C, gives to the agent a the knowledge that D holds". Future work could be devoted to the study of other epistemic separation logics with epistemic actions [Baltag et al. 2006], or updates [Herzig 2013]. Finally, there is a relation of our work with resource-bounded logics of agency. Logics of agency such as ATL [Alur et al. 2002] are very different from dynamic epistemic logics that are essentially logics of observation and that lack agency. In extensions of ATL-like logics one can measure the cost of coalitions performing actions as consumption of resources, and other features model the production of resources. These matters have been investigated in, for example, [Alechina et al. 2017].

2. A Public Announcement Separation Logic

In this section we first present an Epistemic Separation Logic, called ESL, that can be seen as an extension of Boolean BI with a knowledge modality [Courtault et al. 2015] and then we extend the logic with operators for knowledge change, namely public announcements, and then propose a new logic called Public Announcement Separation Logic (PASL).

2.1. *An Epistemic Separation Logic (ESL)*

We assume a finite set of agents A, and a countable set of propositional symbols Prop. The language $\mathcal{L}$ of the Epistemic Separation Logic, denoted ESL, is defined as follows:

$$\varphi ::= p \mid \bot \mid \mathrm{I} \mid \varphi \to \varphi \mid \varphi * \varphi \mid \varphi \mathrel{-\!\!*} \varphi \mid K_a \varphi$$

where a ranges over A and p over Prop.
We can also define the other connectives : $\neg\varphi \equiv \varphi \to \bot$, $\top \equiv \neg\bot$, $\varphi \vee \psi \equiv \neg\varphi \to \psi$, $\varphi \wedge \psi \equiv \neg(\varphi \to \neg\psi)$ and $\widetilde{K}_a \varphi \equiv \neg K_a \neg\varphi$.

Here we consider possible worlds as resources and then we use indifferently the words *possible world* and *resource*.
The *epistemic modality* $K_a\varphi$ means that the agent a knows that φ holds, and the *epistemic modality* $\widetilde{K}_a\varphi$, defined by $\widetilde{K}_a\varphi \equiv \neg K_a\neg\varphi$, means that the agent a considers that φ is possible. Finally the *multiplicative connectives* are the multiplicative conjunction $\varphi * \psi$, meaning that the possible world can be decomposed into two possible sub-worlds such that the first one satisfies φ and the second one satisfies ψ, and the multiplicative implication $\varphi \mathrel{-\!\!*} \psi$ meaning that by adding any possible world that satisfies φ we obtain a possible world that satisfies ψ. We also notice that I is the unit of $*$.
A key point is the mixing of the epistemic modalities and the multiplicative connectives. For example, we can write the formula $\varphi \mathrel{-\!\!*} K_a\psi$ that expresses that any addition of a resource that satisfies φ allows the agent a to obtain the knowledge of ψ.
Let us give now some details about ESL and its semantics [Courtault et al. 2015] before to study its extension with public announcements.

Definition 2.1 (Partial resource monoid). A *partial resource monoid* (PRM) is a structure $\mathcal{R} = (R, \bullet, e)$ such that:
– R is a set of *resources* or *possible worlds* with $e \in R$;
– $\bullet : R \times R \rightharpoonup R$ such that, for all $r_1, r_2, r_3 \in R$, $r_1 \bullet e \downarrow$ and $r_1 \bullet e = r_1$ (neutral element), if $r_1 \bullet r_2 \downarrow$ then $r_2 \bullet r_1 \downarrow$ and $r_1 \bullet r_2 = r_2 \bullet r_1$ (commutativity) and if $r_1 \bullet (r_2 \bullet r_3) \downarrow$ then $(r_1 \bullet r_2) \bullet r_3 \downarrow$ and $r_1 \bullet (r_2 \bullet r_3) = (r_1 \bullet r_2) \bullet r_3$ (associativity).

where $r_1 \bullet r_2 \downarrow$ means "$r_1 \bullet r_2$ is defined" and $r_1 \bullet r_2 \uparrow$ means "$r_1 \bullet r_2$ is undefined".
We denote $\wp(E)$ the powerset of the set E, namely the set of sets built from E. We call e the *unit resource* and $\bullet$ the *resource composition* operator.
Let us define now what is a model and also the related validity relaltion.

Definition 2.2 (Model). A *model* is a triple $\mathcal{M} = (\mathcal{R}, \{\sim_a\}_{a\in A}, V)$ such that:
– $\mathcal{R} = (R, \bullet, e)$ is a PRM;
– For all $a \in A$, $\sim_a \subseteq R \times R$ is an equivalence relation that is, for all $r_1, r_2, r_3 \in R$, $r_1 \sim_a r_1$ (reflexivity), if $r_1 \sim_a r_2$ then $r_2 \sim_a r_1$ (symmetry), if $r_1 \sim_a r_2$ and $r_2 \sim_a r_3$ then $r_1 \sim_a r_3$ (transitivity);.
– $V : \mathsf{Prop} \to \wp(R)$ is a *valuation*.

If we compare these specific models to the models of Epistemic Logic, we observe that the possible worlds are considered as resources, and they can be composed or decomposed by the function $\bullet$. Compared to the BBI models, the partial resource monoids are extended by equivalence relations on resources parametrized by agents.

Definition 2.3 (ESL Forcing relation, validity). Let $\mathcal{M} = (\mathcal{R}, \{\sim_a\}_{a\in A}, V)$ be a model. The forcing relation $\vDash_{\mathcal{M}} \subseteq R \times \mathcal{L}$ is defined by structural induction, for all $r \in R$, as follows:

$r \vDash_{\mathcal{M}} p$	iff	$r \in V(p)$
$r \vDash_{\mathcal{M}} \top$	always	
$r \vDash_{\mathcal{M}} \bot$	never	
$r \vDash_{\mathcal{M}} \mathrm{I}$	iff	$r = e$
$r \vDash_{\mathcal{M}} \neg\varphi$	iff	$r \nvDash_{\mathcal{M}} \varphi$
$r \vDash_{\mathcal{M}} \varphi \wedge \psi$	iff	$r \vDash_{\mathcal{M}} \varphi$ and $r \vDash_{\mathcal{M}} \psi$
$r \vDash_{\mathcal{M}} \varphi \vee \psi$	iff	$r \vDash_{\mathcal{M}} \varphi$ or $r \vDash_{\mathcal{M}} \psi$
$r \vDash_{\mathcal{M}} \varphi \rightarrow \psi$	iff	$r \vDash_{\mathcal{M}} \varphi$ implies $r \vDash_{\mathcal{M}} \psi$
$r \vDash_{\mathcal{M}} \varphi * \psi$	iff	$\exists r_1, r_2 \in R \cdot r_1 \bullet r_2 \downarrow$ and $r = r_1 \bullet r_2$ and $r_1 \vDash_{\mathcal{M}} \varphi$ and $r_2 \vDash_{\mathcal{M}} \psi$
$r \vDash_{\mathcal{M}} \varphi \mathrel{-\!\!*} \psi$	iff	$\forall r' \in R \cdot (r \bullet r' \downarrow$ and $r' \vDash_{\mathcal{M}} \varphi) \Rightarrow r \bullet r' \vDash_{\mathcal{M}} \psi$
$r \vDash_{\mathcal{M}} K_a\varphi$	iff	$\forall r' \in R \cdot r \sim_a r' \Rightarrow r' \vDash_{\mathcal{M}} \varphi$
$r \vDash_{\mathcal{M}} \widetilde{K}_a\varphi$	iff	$\exists r' \in R \cdot r \sim_a r'$ and $r' \vDash_{\mathcal{M}} \varphi$

We say that a formula φ is *valid*, denoted $\vDash \varphi$, if and only if $r \vDash_{\mathcal{M}} \varphi$ for all resources r of all models $\mathcal{M}$.

More details about ESL can be found in [Courtault et al. 2015]. In this paper we study the interest and impact of the addition of public announcements to ESL for modelling and also for proving, in the perpsective of system verification.

2.2. *An Epistemic Separation Logic with Public Announcements*

We aim at extending the language of ESL with the connectives $[\varphi]\psi$ and $\langle\varphi\rangle\psi \equiv \neg[\varphi]\neg\psi$ that are dynamic epistemic modalities of Public Announcement Logic (PAL) [Plaza 1989,van Ditmarsch et al. 2007], $[\varphi]\psi$ meaning that "after the truthful public announcement φ, ψ is true", and $\langle\varphi\rangle\psi$ meaning that "φ can be truthfully announced and ψ is true after it".
The peculiarity of PAL, and of other dynamic epistemic logics, is that this modality is standardly interpreted by a model transformation and not by an internal step in a given model, corresponding to an arrow in a given accessibility relation.
The formula $[\varphi]\psi$ is true in a state of a given model, if and only if on condition that φ is true in that state, in the model restriction to the states where φ is true, the postcondition ψ is true in that state. In PAL terminology, where R is a set of words, $r \models_{\mathcal{M}} [\varphi]\psi$ iff if $r \models_{\mathcal{M}} \varphi$ then $r \models_{\mathcal{M}|\varphi} \psi$ where $\mathcal{M}|\varphi = (R', \{\sim'_a\}_{a\in A}, V')$ such that $R' = \{r \in R \mid r \models_{\mathcal{M}} \varphi\}$, for each $a \in A$, $\sim'_a = \sim_a \cap (R' \times R')$, and for each $p \in P$, $V'(p) = V(p) \cap R'$.

This standard semantics for public announcement logic is unsuitable in our setting, because it does not preserve monoids. For example, given a unit $e \in R$, a public announcement $\neg\mathrm{I}$ will restrict the resource set R of the monoid $\mathcal{R}$ to $R \setminus \{e\}$ that is no longer a monoid. Such restrictions on R cannot preserve the associativity of $\bullet$.
Two alternative semantics for public announcement logic are as follows. In the first approach [Gerbrandy 1999] we do not restrict the domain to worlds where the announcement formula φ is true, but we restrict the accessibility relation (for all agents) to those pairs ending in worlds where φ is true. In a second approach [van Benthem and Liu 2007] we do not restrict the domain but only refine the accessibility relation, i.e., we separate the submodel consisting of the φ worlds from the submodel consisting of the $\neg\varphi$ worlds.

All semantics are equivalent in the sense that in a world satisfying the announcement, the same formulae in the logic are true (they are bisimilar), but the two alternatives have the advantage that the entire domain of the original model is preserved and therefore they preserve monoids. The refinement approach seems most suitable in our setting, as we focus on the incorporation of reliable information, i.e., truthful announcements. We therefore now employ this semantics in our separation logic with announcements.
After this semantic analysis we can define a new epistemic separation logic, called Public Announcement Separation Logic (PASL), by extending the definitions given for ESL in order to deal with the connectives $[\varphi]\psi$ and $\langle\varphi\rangle\psi \equiv \neg[\varphi]\neg\psi$.

We assume a finite set of agents A, and a countable set of propositional symbols Prop. The language $\mathcal{L}$ of PASL is defined as follows:

$$\varphi ::= p \mid \bot \mid \mathrm{I} \mid \varphi \to \varphi \mid \varphi * \varphi \mid \varphi \mathrel{-\!\!*} \varphi \mid K_a\varphi \mid [\varphi]\varphi \mid \langle\varphi\rangle\varphi.$$

where a ranges over A and p over Prop.
From the definition of partial resource monoid and of model we can define the forcing relation for PASL and the related notion of validity.

Definition 2.4 (PASL Forcing relation). Let $\mathcal{M} = (\mathcal{R}, \{\sim_a\}_{a\in A}, V)$ be a model. The forcing relation $\vDash_{\mathcal{M}} \subseteq R \times \mathcal{L}$ of Def. 2.4 is expanded with the following dual inductive clauses for public announcement:

$$\begin{array}{lll} r \vDash_{\mathcal{M}} [\varphi]\psi & \text{iff} & r \vDash_{\mathcal{M}} \varphi \text{ implies } r \vDash_{\mathcal{M}|\varphi} \psi \\ r \vDash_{\mathcal{M}} \langle\varphi\rangle\psi & \text{iff} & r \vDash_{\mathcal{M}} \varphi \text{ and } r \vDash_{\mathcal{M}|\varphi} \psi \end{array}$$

where $\mathcal{M}|\varphi = (\mathcal{R}', \{\sim'_a\}_{a\in A}, V')$, called the *refinement* of $\mathcal{M}$ with φ, is defined by: $\mathcal{R}' = \mathcal{R}$, $\sim'_a = \sim_a \cap \{(r,s) \mid r \vDash_{\mathcal{M}} \varphi \text{ iff } s \vDash_{\mathcal{M}} \varphi\}$ and $V' = V$.

The reader may note the difference with the more standard public announcement semantics given above. Moreover we observe that in the forcing relation there is no interaction between the epistemic aspects and resource aspects: the clauses for $*$ and $\mathrel{-\!\!*}$ do not refer to the equivalence relation that encodes the epistemic modality, and the clauses for knowledge K_a and its dual do not refer to resource composition or decomposition that encode the resource modalities.
In this context we aim at showing that ESL is equally expressive as PASL. For that we first prove the following proposition and two related lemmas.

Proposition 2.5. The following are validities of PASL, namely epistemic separation logic with public announcements.

$$\begin{array}{lll} \langle\varphi\rangle p & \leftrightarrow & (\varphi \wedge p) \\ \langle\varphi\rangle \mathbf{I} & \leftrightarrow & (\varphi \wedge \mathbf{I}) \\ \langle\varphi\rangle(\psi \vee \chi) & \leftrightarrow & (\langle\varphi\rangle\psi \vee \langle\varphi\rangle\chi) \\ \langle\varphi\rangle\neg\psi & \leftrightarrow & (\varphi \wedge \neg\langle\varphi\rangle\psi) \\ \langle\varphi\rangle\widetilde{K}_a\psi & \leftrightarrow & (\varphi \wedge \widetilde{K}_a\langle\varphi\rangle\psi) \\ \langle\varphi\rangle\langle\psi\rangle\chi & \leftrightarrow & \langle\langle\varphi\rangle\psi\rangle\chi \end{array}$$

$$\begin{array}{lll} \langle\varphi\rangle(\psi * \chi) & \leftrightarrow & (\varphi \wedge (\\ & & (\langle\varphi\rangle\psi * \langle\varphi\rangle\chi) \vee \\ & & (\langle\varphi\rangle\psi * \langle\neg\varphi\rangle\chi) \vee \\ & & (\langle\neg\varphi\rangle\psi * \langle\varphi\rangle\chi) \vee \\ & & (\langle\neg\varphi\rangle\psi * \langle\neg\varphi\rangle\chi))) \\ \langle\varphi\rangle(\psi \mathrel{-\!\!*} \chi) & \leftrightarrow & (\varphi \wedge \\ & & \langle\varphi\rangle\psi \mathrel{-\!\!*} (\langle\varphi\rangle\chi \vee \langle\neg\varphi\rangle\chi) \wedge \\ & & \langle\neg\varphi\rangle\psi \mathrel{-\!\!*} (\langle\varphi\rangle\chi \vee \langle\neg\varphi\rangle\chi)) \end{array}$$

Proof. For all but the last two schemata, this is straightforward and proceeds as in public announcement logic. We recall that there is no interaction between the epistemic modalities K_a and the separation logic primitives of decomposition $*$ and composition $-\!\!*$. (The identity can be treated as just another propositional variable; **I** is merely a designated variable with a unique interpretation.) For the last two schemata involving the interaction between public announcement and the multiplicative connectives $*$ and $-\!\!*$, we refer to Lemma 2.6 and 2.7 below. $\square$

Lemma 2.6. A validity of the logic is

$$\langle\varphi\rangle(\psi*\chi) \leftrightarrow (\varphi \wedge (((\langle\varphi\rangle\psi*\langle\varphi\rangle\chi) \vee (\langle\varphi\rangle\psi*\langle\neg\varphi\rangle\chi) \vee (\langle\neg\varphi\rangle\psi*\langle\varphi\rangle\chi) \vee (\langle\neg\varphi\rangle\psi*\langle\neg\varphi\rangle\chi)))$$

Proof. Let $\mathcal{M} = (\mathcal{R}, \{\sim_a\}_{a\in A}, V)$ and $r \in \mathcal{R}$ be given.

$r \vDash_{\mathcal{M}} \langle\varphi\rangle(\psi*\chi)$ iff
$r \vDash_{\mathcal{M}} \varphi$ and $r \vDash_{\mathcal{M}|\varphi} \psi*\chi$ iff
$r \vDash_{\mathcal{M}} \varphi$ and $\exists r_1, r_2 \in R \cdot r_1 \bullet r_2 \downarrow$ and $r = r_1 \bullet r_2$ and $r_1 \vDash_{\mathcal{M}|\varphi} \psi$ and $r_2 \vDash_{\mathcal{M}|\varphi} \chi$ iff
$r \vDash_{\mathcal{M}} \varphi$ and $\exists r_1, r_2 \in R \cdot r_1 \bullet r_2 \downarrow$ and $r = r_1 \bullet r_2$ and $(r_1 \vDash_{\mathcal{M}} \varphi$ or $r_1 \vDash_{\mathcal{M}} \neg\varphi)$ and $r_1 \vDash_{\mathcal{M}|\varphi} \psi$ and $(r_2 \vDash_{\mathcal{M}} \varphi$ or $r_2 \vDash_{\mathcal{M}} \neg\varphi)$ and $r_2 \vDash_{\mathcal{M}|\varphi} \chi$

We now observe that $\mathcal{M}|\varphi = \mathcal{M}|\neg\varphi$ (the refinement with the denotation of φ 'refines' the domain into a φ and a $\neg\varphi$ part; we remind the reader that our public announcement is *not* a model restriction) and that
$(r_1 \vDash_{\mathcal{M}} \varphi$ or $r_1 \vDash_{\mathcal{M}} \neg\varphi)$ and $r_1 \vDash_{\mathcal{M}|\varphi} \psi$ iff
$(r_1 \vDash_{\mathcal{M}} \varphi$ and $r_1 \vDash_{\mathcal{M}|\varphi} \psi)$ or $(r_1 \vDash_{\mathcal{M}} \neg\varphi$ and $r_1 \vDash_{\mathcal{M}|\varphi} \psi)$ iff
$r_1 \vDash_{\mathcal{M}} \langle\varphi\rangle\psi$ or $r_1 \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\psi$

and similarly $(r_2 \vDash_{\mathcal{M}} \varphi$ or $r_2 \vDash_{\mathcal{M}} \neg\varphi)$ and $r_2 \vDash_{\mathcal{M}|\varphi} \chi$ is equivalent to $r_2 \vDash_{\mathcal{M}} \langle\varphi\rangle\chi$ or $r_2 \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\chi$. The above expression is therefore equivalent to
$r \vDash_{\mathcal{M}} \varphi$ and $\exists r_1, r_2 \in R \cdot r_1 \bullet r_2 \downarrow$ and $r = r_1 \bullet r_2$ and $(r_1 \vDash_{\mathcal{M}} \langle\varphi\rangle\psi$ or $r_1 \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\psi)$ and $(r_2 \vDash_{\mathcal{M}} \langle\varphi\rangle\chi$ or $r_2 \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\chi)$

By boolean manipulations we get from this the equivalent
$r \vDash_{\mathcal{M}} \varphi$ and $\exists r_1, r_2 \in R \cdot r_1 \bullet r_2 \downarrow$ and $r = r_1 \bullet r_2$ and $(r_1 \vDash_{\mathcal{M}} \langle\varphi\rangle\psi$ and $r_2 \vDash_{\mathcal{M}} \langle\varphi\rangle\chi)$ or $(r_1 \vDash_{\mathcal{M}} \langle\varphi\rangle\psi$ and $r_2 \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\chi)$ or $(r_1 \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\psi$ and $r_2 \vDash_{\mathcal{M}} \langle\varphi\rangle\chi)$ or $(r_1 \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\psi$ and $r_2 \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\chi)$
and by further boolean manipulations and the distribution of the existential quantifier over the disjunction we get
$r \vDash_{\mathcal{M}} \varphi$ and
$\exists r_1, r_2 \in R \cdot r_1 \bullet r_2 \downarrow$ and $r = r_1 \bullet r_2$ and $r_1 \vDash_{\mathcal{M}} \langle\varphi\rangle\psi$ and $r_2 \vDash_{\mathcal{M}} \langle\varphi\rangle\chi$ or
$\exists r_1, r_2 \in R \cdot r_1 \bullet r_2 \downarrow$ and $r = r_1 \bullet r_2$ and $r_1 \vDash_{\mathcal{M}} \langle\varphi\rangle\psi$ and $r_2 \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\chi$ or
$\exists r_1, r_2 \in R \cdot r_1 \bullet r_2 \downarrow$ and $r = r_1 \bullet r_2$ and $r_1 \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\psi$ and $r_2 \vDash_{\mathcal{M}} \langle\varphi\rangle\chi$ or
$\exists r_1, r_2 \in R \cdot r_1 \bullet r_2 \downarrow$ and $r = r_1 \bullet r_2$ and $r_1 \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\psi$ and $r_2 \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\chi$

By the semantics of $*$ this is equivalent to
$r \vDash_{\mathcal{M}} \varphi$ and $r \vDash_{\mathcal{M}} \langle\varphi\rangle\psi * \langle\varphi\rangle\chi$ or $r \vDash_{\mathcal{M}} \langle\varphi\rangle\psi * \langle\neg\varphi\rangle\chi$ or $r \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\psi * \langle\varphi\rangle\chi$ or $r \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\psi * \langle\neg\varphi\rangle\chi$

and this finally delivers us the desired
$r \vDash_{\mathcal{M}} \varphi \wedge (((\langle\varphi\rangle\psi * \langle\varphi\rangle\chi) \vee ((\langle\varphi\rangle\psi * \langle\neg\varphi\rangle\chi) \vee ((\langle\neg\varphi\rangle\psi * \langle\varphi\rangle\chi) \vee ((\langle\neg\varphi\rangle\psi * \langle\neg\varphi\rangle\chi)$).
As all these steps were equivalences, and as $\mathcal{M}$ and r were arbitrary, we are done. □

Lemma 2.7. A validity of the logic is

$$\langle\varphi\rangle(\psi -\!\!* \chi) \leftrightarrow (\varphi \wedge \langle\varphi\rangle\psi -\!\!* (\langle\varphi\rangle\chi \vee \langle\neg\varphi\rangle\chi) \wedge \langle\neg\varphi\rangle\psi -\!\!* (\langle\varphi\rangle\chi \vee \langle\neg\varphi\rangle\chi))$$

Proof. Let $\mathcal{M} = (\mathcal{R}, \{\sim_a\}_{a\in A}, V)$ and $r \in \mathcal{R}$ be given.

$r \vDash_{\mathcal{M}} \langle\varphi\rangle(\psi -\!\!* \chi)$ iff
$r \vDash_{\mathcal{M}} \varphi$ and $r \vDash_{\mathcal{M}|\varphi} \psi -\!\!* \chi$ iff
$r \vDash_{\mathcal{M}} \varphi$ and $\forall r' \in R \cdot (r \bullet r' \downarrow$ and $r' \vDash_{\mathcal{M}|\varphi} \psi) \Rightarrow r \bullet r' \vDash_{\mathcal{M}|\varphi} \chi$ iff
$r \vDash_{\mathcal{M}} \varphi$ and $\forall r' \in R \cdot r \bullet r' \downarrow \Rightarrow (r' \vDash_{\mathcal{M}|\varphi} \psi \Rightarrow r \bullet r' \vDash_{\mathcal{M}|\varphi} \chi)$ iff
$r \vDash_{\mathcal{M}} \varphi$ and $\forall r' \in R \cdot r \bullet r' \downarrow \Rightarrow$ ($(r' \vDash_{\mathcal{M}} \varphi$ or $r' \vDash_{\mathcal{M}} \neg\varphi)$ and $r' \vDash_{\mathcal{M}|\varphi} \psi \Rightarrow ((r \bullet r' \vDash_{\mathcal{M}} \varphi$ or
$r \bullet r' \vDash_{\mathcal{M}} \neg\varphi)$ and $r \bullet r' \vDash_{\mathcal{M}|\varphi} \chi)$) iff
$r \vDash_{\mathcal{M}} \varphi$ and $\forall r' \in R \cdot r \bullet r' \downarrow \Rightarrow$ ($((r' \vDash_{\mathcal{M}} \varphi$ and $r' \vDash_{\mathcal{M}|\varphi} \psi)$ or $(r' \vDash_{\mathcal{M}} \neg\varphi$ and $r' \vDash_{\mathcal{M}|\varphi} \psi)) \Rightarrow$
$((r \bullet r' \vDash_{\mathcal{M}} \varphi$ and $r \bullet r' \vDash_{\mathcal{M}|\varphi} \chi)$ or $(r \bullet r' \vDash_{\mathcal{M}} \neg\varphi$ and $r \bullet r' \vDash_{\mathcal{M}|\varphi} \chi))$) iff
$r \vDash_{\mathcal{M}} \varphi$ and $\forall r' \in R \cdot r \bullet r' \downarrow \Rightarrow$
$(r' \vDash_{\mathcal{M}} \langle\varphi\rangle\psi$ or $r' \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\psi) \Rightarrow (r \bullet r' \vDash_{\mathcal{M}} \langle\varphi\rangle\chi$ or $r \bullet r' \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\chi)$ iff
$r \vDash_{\mathcal{M}} \varphi$ and $\forall r' \in R \cdot r \bullet r' \downarrow \Rightarrow (r' \vDash_{\mathcal{M}} \langle\varphi\rangle\psi \Rightarrow (r \bullet r' \vDash_{\mathcal{M}} \langle\varphi\rangle\chi$ or $r \bullet r' \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\chi)$ and
$(r' \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\psi) \Rightarrow (r \bullet r' \vDash_{\mathcal{M}} \langle\varphi\rangle\chi$ or $r \bullet r' \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\chi)$ (∗) iff
$r \vDash_{\mathcal{M}} \varphi$ and
$\forall r' \in R \cdot r \bullet r' \downarrow \Rightarrow (r' \vDash_{\mathcal{M}} \langle\varphi\rangle\psi \Rightarrow (r \bullet r' \vDash_{\mathcal{M}} \langle\varphi\rangle\chi$ or $r \bullet r' \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\chi))$ and
$\forall r' \in R \cdot r \bullet r' \downarrow \Rightarrow (r' \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\psi \Rightarrow (r \bullet r' \vDash_{\mathcal{M}} \langle\varphi\rangle\chi$ or $r \bullet r' \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\chi))$ iff
$r \vDash_{\mathcal{M}} \varphi$ and
$\forall r' \in R \cdot r \bullet r' \downarrow \Rightarrow (r' \vDash_{\mathcal{M}} \langle\varphi\rangle\psi \Rightarrow r \bullet r' \vDash_{\mathcal{M}} \langle\varphi\rangle\chi \vee \langle\neg\varphi\rangle\chi)$ and
$\forall r' \in R \cdot r \bullet r' \downarrow \Rightarrow (r' \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\psi \Rightarrow r \bullet r' \vDash_{\mathcal{M}} \langle\varphi\rangle\chi \vee \langle\neg\varphi\rangle\chi)$ iff
$r \vDash_{\mathcal{M}} \varphi$ and $r \vDash_{\mathcal{M}} \langle\varphi\rangle\psi -\!\!* (\langle\varphi\rangle\chi \vee \langle\neg\varphi\rangle\chi)$ and $r \vDash_{\mathcal{M}} \langle\neg\varphi\rangle\psi -\!\!* (\langle\varphi\rangle\chi \vee \langle\neg\varphi\rangle\chi)$ iff
$r \vDash_{\mathcal{M}} \varphi \wedge (\langle\varphi\rangle\psi -\!\!* (\langle\varphi\rangle\chi \vee \langle\neg\varphi\rangle\chi)) \wedge (\langle\neg\varphi\rangle\psi -\!\!* (\langle\varphi\rangle\chi \vee \langle\neg\varphi\rangle\chi))$

(*): by distribution of universal quantifier over conjunction. □

Proposition 2.8. Epistemic separation logic (ESL) is equally expressive as epistemic separation logic with public announcements (PASL)

Proof. The proof is along the standard lines for the elimination of dynamic modalities in dynamic epistemic logics without common knowledge [Plaza 1989,van Ditmarsch et al. 2007]. Given a formula in the logic with public announcements, we can iteratively push all announcements further into the formula by one of the equivalences above (that is, by a rewriting technique), until we finally reach a(n) (equivalent) formula wherein the announcement is in front of a variable, so that it can be eliminated. Therefore, every formula with public announcements is logically equivalent to one without public announcements. □

As ESL does not contain dynamic modalities, obviously the update expressivity of PASL is larger than that of ESL. On the other hand, the logics are equally expressive but the advantage of PASL is then the availability of more succinct descriptions of systems behaviour. Any PASL specification that after some announcements a certain property holds is equivalent to some ESL formula, i.e., without announcements, but at the risk of the latter being exponentially longer than the former. We have not investigated this for PASL, but it is known that the addition of dynamics to similar logics tends to make them more succinct [Lutz 2006,French et al 2013].

3. Modelling with Public Announcement Separation Logic

First we develop an example that emphasizes some key points about modelling with PASL. We consider two agents that enter in a library to borrow books. We suppose that they are not allowed to take out more than two books (only zero, one or two books) and they must tell the book references to the librarian who will fetch their books. We also suppose that the books asked by the agents are always available and that each agent does not know which books and how many books are asked by the other. The librarian says to the agents: “Before telling me the book references I would like to say that I cannot carry more than two books. Could you tell me, at first, if I will be able to carry all the books that you want or if I need to use a book trolley ?”.

As a first step, we build a model of this situation with PASL. We define the set of agents $A = \{A_1, A_2\}$, where A_i is the i^{th} agent and a PRM that deals with the possible worlds $\mathcal{R} = (R, \bullet, e)$. Then we define the set of resources $R = \{(i,j) \mid i,j \in \{0,1,2\}\}$, where (i,j) encodes “the agent A_1 wants i books and the agent A_2 wants j books”, and we recall that an agent cannot borrow more than two books. Thereby, for instance, $(2,0)$ represents A_1 that wants two books and A_2 that wants zero book.
The resource composition $\bullet$ is defined by:

$$(i_1, j_1) \bullet (i_2, j_2) = \begin{cases} \uparrow & \text{if } i_1 + i_2 > 2 \text{ or } j_1 + j_2 > 2 \\ (i_1 + i_2, j_1 + j_2) & \text{otherwise} \end{cases}$$

We recall that $\uparrow$ means “is not defined” and we note that $(0,0)$ is the unit of resource composition and then $e = (0,0)$.

Let us now illustrate the resource composition. We assume that A_1 wants to borrow one book and the other agent A_2 wants no book, then we represent the global borrow request by the resource, or possible world, $(1,0)$. Now, if A_2 wants two more books, then we have the final borrow request $(1,0) \bullet (0,2) = (1,2)$. Moreover, if A_2 wants one more book then it is not allowed: we have $(1,2) \bullet (0,1) \uparrow$, that expresses that A_2 cannot borrow more than two books.
Now, we have to build a model $\mathcal{M} = (\mathcal{R}, \{\sim_a\}_{a \in A}, V)$ and then we define two equivalence relations, that are $\sim_{A_1}$ and $\sim_{A_2}$. For instance, we expect $(1,0) \sim_{A_2} (2,0)$ because if A_2 wants no book then, as A_2 has no information about how many books are wanted by A_1 and as he has only information about how many books he wants, then he must consider, from his point of view, that A_1 might want one book or A_1 might want two books. On the other hand, we also expect to have, for instance, $(1,0) \not\sim_{A_2} (1,1)$, because it is not consistent, from the point of view of A_2, that he wants no book and one book.
Therefore, we give the following definitions, for all $i_1, i_2, j_1, j_2 \in \{0,1,2\}$:

$$\begin{array}{lll} (i_1, j_1) \sim_{A_1} (i_2, j_2) & \text{iff} & i_1 = i_2 \\ (i_1, j_1) \sim_{A_2} (i_2, j_2) & \text{iff} & j_1 = j_2 \end{array}$$

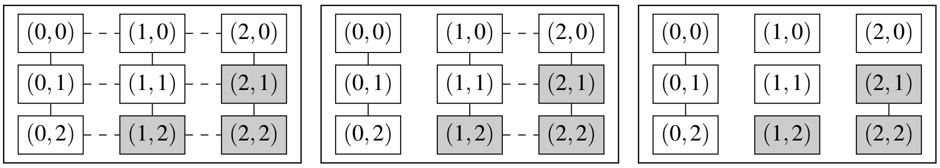


Fig. 1. On the left, the initial model of knowledge. Dashed links - - - represent the relation $\sim_{A_2}$, solid links — represent the relation $\sim_{A_1}$. We assume reflexivity and transitivity. Grey means "cannot be carried". In the middle, the update of the model after the first announcement $K_{A_1}\neg\mathrm{I}$. On the right, the model $\mathcal{M}|\Upsilon_1|\Upsilon_2|\Upsilon_3|\Upsilon_4$ after all four announcements.

Finally, we consider the set of propositional symbols $\mathsf{Prop} = \{P_1, P_2, C\}$ and the valuation V, such that $V(P_1) = \{(1,0)\}$, $V(P_2) = \{(0,1)\}$ and $V(C) = \{(i,j) \mid i+j \leqslant 2\}$. Thus we have $r \in V(P_i)$ if and only if r is the borrow such that the agent A_i wants one and only one book and the other agent wants zero book and $r \in V(C)$ means that the librarian can carry the books of r (the agents want at maximum two books).

A graphical representation of our model is given in Fig. 1, where grey vertices correspond to requests which do not satisfy C.
After the construction of the model of Fig. 1, let us illustrate the use of PASL connectives in our model. Concerning propositional symbols, we have for instance $(0,1) \vDash_{\mathcal{M}} P_2$, because $(0,1) \in V(P_2)$, which expresses that only one book is wanted and this book is wanted by A_2. But, we have $(0,2) \nvDash_{\mathcal{M}} P_2$ and $(1,1) \nvDash_{\mathcal{M}} P_2$. Concerning the propositional symbol C, we have for instance $(1,1) \vDash_{\mathcal{M}} C$ which expresses that the librarian can carry the two books asked by the agents, but $(1,2) \nvDash_{\mathcal{M}} C$ that means that the librarian cannot carry the books (because the agents want more than two books).
PASL can express properties on agent knowledge. For instance, we have $(0,1) \vDash_{\mathcal{M}} K_{A_1}C$, because for all $r \in R$ such that $(0,1) \sim_{A_1} r$, we have $r \vDash_{\mathcal{M}} C$. It means that if we consider that A_1 wants no book and A_2 wants one book, then the agent A_1 knows that the librarian can carry the books. Concerning the modality $\widetilde{K}_a$ we have $(1,2) \vDash_{\mathcal{M}} \widetilde{K}_{A_1}C$, because $(1,2) \sim_{A_1} (1,1)$ and $(1,1) \vDash_{\mathcal{M}} C$. It means that if A_1 wants one book and A_2 wants two books then A_1 considers that it is possible that the librarian can carry the books.
PASL can also express sharing and separation properties. Concerning the formula I, we have $r \vDash_{\mathcal{M}} \mathrm{I}$ iff $r = e = (0,0)$. In other words the formula I expresses that the agents want no book. About sharing and separation expressed in ESL, as $(0,0) \vDash_{\mathcal{M}} K_{A_1}C$ and $(0,0) \vDash_{\mathcal{M}} K_{A_2}C$ then we have $(0,0) \vDash_{\mathcal{M}} K_{A_1}C \wedge K_{A_2}C$. The conjunction $\wedge$ expresses sharing such that $K_{A_1}C$ and $K_{A_2}C$ share the resource $(0,0)$. The other conjunction $*$ expresses separation. As $(2,0) = (1,0) \bullet (1,0)$ and $(1,0) \vDash_{\mathcal{M}} P_1$ and $(1,0) \vDash_{\mathcal{M}} P_1$ then $(2,0) \vDash_{\mathcal{M}} P_1 * P_1$. This is a separation property because $(2,0)$ is separated (or decomposed) into two sub-resources. We remark that $P_1 * P_1$ means that A_1 wants two books (and the other agent A_1 wants zero book) and the connective $*$ allows us to count resources. For instance, $P_1 * P_2 * P_2$ means that A_1 wants one book and A_2 wants two books. The multiplicative implication $-\!\!*$ allows us to express a property on the resource obtained after the addition of another resource. For instance $(1,1) \vDash_{\mathcal{M}} P_1 -\!\!* \neg C$, because if we add a resource that satisfies P_1 to the resource $(1,1)$ then we obtain a resource that satisfies $\neg C$. Indeed we only have $(1,0) \vDash_{\mathcal{M}} P_1$ and then $(1,1) \bullet (1,0) = (2,1)$ and $(2,1) \vDash_{\mathcal{M}} \neg C$.

Then we have $(1,1) \vDash_{\mathcal{M}} P_1 \twoheadrightarrow \neg C$, that means that if A_1 and A_2 want one book then if A_1 wants one more book then the librarian cannot carry the books.

After the librarian asks to the agents if he will be able to carry the wanted books, we suppose that the agents have the following discussion:
1. A_1: "I know that we do not want zero book."
2. A_2: "I know that I want at least one book, and A_1 also wants at least one book."
3. A_1: "I know that I am allowed to borrow one more book."
4. A_2: "I know that you can carry our books. Moreover, I also know that each of us wants one book."

The previous sentences numbered by i are public announcements, which will be denoted Υ_i. We now show the evolution of the model of Fig. 1 after each announcement.
For the sake of readability, we only depict the connected submodel in our interest and not the disconnected remaining parts of the model (we recall the reader that our public announcement semantics refines the model and does not restrict the model).

Firstly, A_1 says (announces) that he knows that the agents do not want no book, which is expressed by the formula $\Upsilon_1 = K_{A_1}\neg \mathrm{I}$. We observe that we have, $(i,j) \vDash_{\mathcal{M}} K_{A_1}\neg \mathrm{I}$ if and only if $(i,j) \not\sim_{A_1} (0,0)$. Then the update of our model by the public announcement $K_{A_1}\neg \mathrm{I}$ is the model $\mathcal{M}|K_{A_1}\neg \mathrm{I}$ which is given in Fig. 1.
Starting from the model $\mathcal{M}|K_{A_1}\neg \mathrm{I}$ which is given in Fig. 1 and assuming that the agents never lie, the worlds $(0,j)$, where $j \in \{0,1,2\}$, cannot be the solution of our problem because these words do not force the public announcement. We call "solution of our problem" any world that allows the agents to do the announcements without lying. Thus, the solution is one of the possible worlds of Fig. 2. We will continue to focus on the connected submodel only.
Then, A_2 announces that he knows that A_1 wants at least one book, and also himself wants at least one book. Such property is expressed by the formula $\Upsilon_2 = K_{A_2}((P_1 * \top) \wedge (P_2 * \top))$. We have $(i,j) \vDash_{\mathcal{M}|\Upsilon_1} K_{A_2}((P_1 * \top) \wedge (P_2 * \top))$ iff $i \geq 1$ and $j \geq 1$. Then, focusing on the possible worlds satisfying the formula, the solution is one of the resources of Fig. 3. A_1 announces that he knows that he is allowed to borrow one more book, which is captured by the formula $\Upsilon_3 = K_{A_1}\neg(P_1 \twoheadrightarrow \perp)$. Indeed, we have $(i,j) \vDash_{\mathcal{M}|\Upsilon_1|\Upsilon_2} P_1 \twoheadrightarrow \perp$ if and only if for all $r \in R$ such that $(i,j) \bullet r \downarrow$ and $r \vDash_{\mathcal{M}|\Upsilon_1|\Upsilon_2} P_1$, we have $(i,j) \bullet r \vDash_{\mathcal{M}|\Upsilon_1|\Upsilon_2} \perp$. As r can only be $(1,0)$ (because $r \vDash_{\mathcal{M}|\Upsilon_1|\Upsilon_2} P_1$) and no resource satisfies $\perp$, we necessarily have $(i,j) \bullet (1,0) \uparrow$, that means that A_1 cannot borrow one more book. Then the negation ($\neg$) of the formula $(P_1 \twoheadrightarrow \perp)$ means A_1 can borrow one more book. Finally, ignoring all possible worlds that do not satisfy the formula $K_{A_1}\neg(P_1 \twoheadrightarrow \perp)$, we obtain the worlds of Fig. 4.
Finally, A_2 says that he knows that the librarian can carry the books. The only possible world which satisfies the formula $K_{A_2}C$ is $(1,1)$, which is the solution of our problem: A_1 wants one book and A_2 wants also one book. Moreover, A_2 knows it, that is expressed by $K_{A_2}(C \wedge (P_1 * P_2))$. Considering this last sentence as a public announcement ($\Upsilon_4 = K_{A_2}(C \wedge (P_1 * P_2))$), and ignoring the worlds that do not satisfy it, we obtain the world of Fig. 5. We also can write $(1,1) \vDash_{\mathcal{M}} \langle\Upsilon_1\rangle\langle\Upsilon_2\rangle\langle\Upsilon_3\rangle K_{A_2}(C \wedge (P_1 * P_2))$, that expresses that after all announcements, A_2 knows that the librarian can carry the books and also knows the quantity of books wanted being each agent.

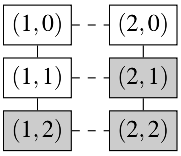


Fig. 2.

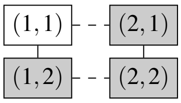


Fig. 3.

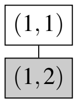


Fig. 4.

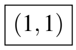


Fig. 5.

We remark that $(1,1)$ is the only world satisfying the formula and the public announcements are expressed using $\langle \Upsilon_i \rangle$ rather than $[\Upsilon_i]$ because they are all true. We assume that the agents are in a true and fair view.

In order to complete this study about expressivity, let us now reason once more about the entire model and not about the situation $(1,1)$. For that we show how we can combine epistemic and separating connectives and then to provide new modalities that allow us to express particular properties.
– $K_a(\varphi * \psi)$, that means that the agent a knows that the resource (the possible world) can be decomposed into two sub-resources that respectively satisfy φ and ψ. Back to the example, $K_{A_1}(P_1 * P_1 * P_2)$ expresses that A_1 knows that he wants two books and A_2 wants one book.
– $K_a(\varphi -\!\!* \psi)$, that means that the agent a knows that by the addition of a resource satisfying φ one obtains a resource satisfying ψ. Back to the example, $K_{A_1}((P_1 \vee P_2) -\!\!* \neg C)$ expresses that A_1 knows that if an agent orders one more book then the librarian cannot carry the books.
– $\varphi * K_a\psi$, that means that without a resource satisfying φ, the agent a could have the knowledge that ψ holds. Back to the example, $P_2 * K_{A_2}C$ expresses that wanting one book less, the agent A_2 gets the knowledge that the librarian can carry the books.
– $\varphi -\!\!* K_a\psi$, that means that the addition of a resource satisfying φ allows the agent a to obtain the knowledge that ψ holds. Back to the example, $P_1 -\!\!* K_{A_1}\neg C$ expresses that choosing to borrow one more book gives to A_1 the knowledge that the librarian cannot carry the books.

We remark that the two last expressions allow us to express a property that involves a kind of change of mind, namely “if the agent wants one book less” and “if the agent chooses to borrow one more book”. The use of such formulae that can be seen as new epistemic modalities will be studied in next works.

4. A Tableaux Calculus for PASL

In this section, we present a tableaux calculus for PASL, designed in the spirit of the tableaux calculus for BI and BBI [Galmiche et al. 2005,Larchey-Wendling 2016], with the extraction of a countermodel in case of non-validity of a formula. Compared to other calculi like ESL [Courtault et al. 2015], we have to deal with public announcements and then we introduce new specific constraints and new rules. An original point is that constraints are decorated with stacks of formulas, knowing that for PAL they only decorate formulas.
Our treatment of announcements has some similarities with what is done in tableaux calculi for PAL [Balbiani et al. 2010]: book-keeping for finite lists of announcements plays an important role, and despite the refinement semantics for announcements in our paper we can maintain the

splitting rules for announcements (as, after all, also in our alternative semantics the truth of the announcement formula is a condition for execution).

4.1. *Labels and constraints*

We first introduce labels and constraints that respectively correspond to resources and the equality and the equivalence relations on resources and agents, including annoucements.

Let us consider an infinite countable set of (resource) constants $\gamma_r = \{c_1, c_2, \ldots\}$. We denote $\oplus$ the concatenation of lists and $[\![]\!]$ the empty list. For example, we have $[\![e_1; e_2]\!] \oplus [\![e_2; e_3]\!] = [\![e_1; e_2; e_2; e_3]\!]$. We also denote $\mathfrak{L}_E$ the set of all lists built over E.

Definition 4.1 (Resource labels). A *resource label* is a word built on γ_r where the order of letters is not taken into account, i.e. a finite multiset of γ_r elements.

We denote L_r the set of all *resource labels* built on γ_r. The composition of resource labels is denoted multiplicatively and ε is the empty word. For instance, xy is the composition of the resource labels x and y. We say that x is a *resource sub-label* of y if and only if there exists z such that $xz = y$. The set of resource sub-labels of x is denoted $\mathcal{E}(x)$.

Definition 4.2 (Constraints). A *resource constraint* is an expression of the form $x \simeq y$ where x and y are resource labels.

An *agent constraint* is an expression of the form $x \coloneqq^{\mu}_{u} y$, where x and y are resource labels, u belongs to the set of agents A and μ is a finite list of formulae of $\mathcal{L}$.

An *announcement constraint* is an expression of the form $\triangleright\mu$, where μ is a finite list of formulae of $\mathcal{L}$.

We call *set of constraints* any set $\mathcal{C}$ that contains resource constraints, agent constraints and announcement constraints. For instance, $\mathcal{C} = \{c_1 \simeq c_2, c_2 \simeq c_3, c_4 \coloneqq^{[\![P;P\wedge Q]\!]}_{b} c_1\}$ is a set of constraints.

Definition 4.3 (Domain). Let $\mathcal{C}$ be a set of constraints. The (resource) *domain* of $\mathcal{C}$ is the set of all resource sub-labels that appear in $\mathcal{C}$, that is:

$$\mathcal{D}_r(\mathcal{C}) = \bigcup_{x \simeq y \in \mathcal{C}} (\mathcal{E}(x) \cup \mathcal{E}(y)) \;\cup \bigcup_{x \coloneqq^{\mu}_{u} y \in \mathcal{C}} (\mathcal{E}(x) \cup \mathcal{E}(y))$$

The announcement *domain* of $\mathcal{C}$ is the set of all list of formulae that appear in $\mathcal{C}$:

$$\mathcal{D}_a(\mathcal{C}) = \bigcup_{x \coloneqq^{\mu}_{u} y \in \mathcal{C}} \{\mu\} \;\cup \bigcup_{\triangleright\mu \in \mathcal{C}} \{\mu\}$$

Definition 4.4 (Alphabet). Let $\mathcal{C}$ be a set of constraints. The (resource) *alphabet* of $\mathcal{C}$ is the set of resource constants that appear in $\mathcal{C}$. In particular, $\mathcal{A}_r(\mathcal{C}) = \gamma_r \cap \mathcal{D}_r(\mathcal{C})$.

Now we introduce rules for constraint closure that allow us to capture the properties of the models into the calculus.

Definition 4.5 (Closure of constraints). Let $\mathcal{C}$ be a set of constraints. The closure of $\mathcal{C}$, denoted $\overline{\mathcal{C}}$, is the least relation closed under the rules of Fig. 6 such that $\mathcal{C} \subseteq \overline{\mathcal{C}}$.

Rules for resource constraints

$$\frac{}{\varepsilon \simeq \varepsilon}\langle \varepsilon \rangle \qquad \frac{x \simeq y}{y \simeq x}\langle s_r \rangle \qquad \frac{xy \simeq xy}{x \simeq x}\langle d_r \rangle \qquad \frac{x \simeq y \qquad y \simeq z}{x \simeq z}\langle t_r \rangle$$

$$\frac{x \simeq y \qquad yk \simeq yk}{xk \simeq yk}\langle c_r \rangle \qquad \frac{x =_u^{\mu} y}{x \simeq x}\langle k_r \rangle$$

Rules for agent constraints

$$\frac{x \simeq x}{x =_v^{\llbracket\rrbracket} x}\langle r_a \rangle \qquad \frac{x =_u^{\llbracket\rrbracket} y}{y =_u^{\llbracket\rrbracket} x}\langle s_a \rangle \qquad \frac{x =_u^{\llbracket\rrbracket} y \qquad y =_u^{\llbracket\rrbracket} z}{x =_u^{\llbracket\rrbracket} z}\langle t_a \rangle$$

$$\frac{x =_u^{\llbracket\rrbracket} y \qquad x \simeq k}{k =_u^{\llbracket\rrbracket} y}\langle k_a \rangle \qquad \frac{x =_u^{\llbracket \psi_1;\ldots;\psi_k \rrbracket} y}{x =_u^{\llbracket \psi_1;\ldots;\psi_{k-1} \rrbracket} y}\langle p_a \rangle$$

Rule for announcement constraints

$$\frac{x =_u^{\mu} y}{\rhd \mu}\langle a_n \rangle$$

Fig. 6. Rules for constraint closure, for all $v \in A$

There are six rules ($\langle \varepsilon \rangle$, $\langle s_r \rangle$, $\langle d_r \rangle$, $\langle t_r \rangle$, $\langle c_r \rangle$ and $\langle k_r \rangle$) that produce resource constraints, five rules ($\langle r_a \rangle$, $\langle s_a \rangle$, $\langle t_a \rangle$, $\langle k_a \rangle$ and $\langle p_a \rangle$) that produce agent constraints and one rule that ($\langle a_n \rangle$) produces announcement constraints.
We note that v, introduced in the rule $\langle r_a \rangle$, must belong to the set of agents A. For instance, if $\mathcal{C} = \{c_1 \simeq c_2, c_2 \simeq c_3, c_1 =_b^{\llbracket P*Q;P\vee R \rrbracket} c_4\}$, we have $c_3 =_b^{\llbracket\rrbracket} c_4 \in \overline{\mathcal{C}}$ because of the following proof:

$$\dfrac{\dfrac{\dfrac{c_1 =_b^{\llbracket P*Q;P\vee R \rrbracket} c_4 \qquad \dfrac{c_1 \simeq c_2 \qquad c_2 \simeq c_3}{c_1 \simeq c_3}\langle t_r \rangle}{c_3 =_b^{\llbracket P*Q;P\vee R \rrbracket} c_4}\langle k_a \rangle}{c_3 =_b^{\llbracket P*Q \rrbracket} c_4}\langle p_a \rangle}{c_3 =_b^{\llbracket\rrbracket} c_4}\langle p_a \rangle$$

Let us remark that $\rhd\mu$ iff $\mu \in \mathcal{D}_a(\overline{\mathcal{C}})$. In fact, if one wants to introduce in the calculus the annoucement list μ then it is not possible to add $x =_u^{\mu} x$ in the set of constraints, because when $A = \emptyset$ it is not possible to find such a u.

Proposition 4.6. The following rules can be derived from the rules of constraint closure:

$$\frac{xk \simeq y}{x \simeq x}\langle p_l \rangle \qquad \frac{x \simeq yk}{y \simeq y}\langle p_r \rangle \qquad \frac{xk =_u^{\llbracket\rrbracket} y}{x \simeq x}\langle q_l \rangle \qquad \frac{x =_u^{\llbracket\rrbracket} yk}{y \simeq y}\langle q_r \rangle$$

$$\frac{x =_u^{\llbracket\rrbracket} y \qquad x \simeq x' \qquad y \simeq y'}{x' =_u^{\llbracket\rrbracket} y'}\langle w_a \rangle$$

Proof. We provide the following deduction trees

$$\dfrac{\dfrac{xk \simeq y \qquad \dfrac{xk \simeq y}{y \simeq xk}\langle s_r \rangle}{xk \simeq xk}\langle t_r \rangle}{x \simeq x}\langle d_r \rangle \qquad \dfrac{\dfrac{x \simeq yk}{yk \simeq x}\langle s_r \rangle}{y \simeq y}\langle p_l \rangle \qquad \dfrac{\dfrac{xk =_u^{\llbracket\rrbracket} y}{xk \simeq xk}\langle k_r \rangle}{x \simeq x}\langle d_r \rangle \qquad \dfrac{\dfrac{x =_u^{\llbracket\rrbracket} yk}{yk =_u^{\llbracket\rrbracket} x}\langle s_a \rangle}{y \simeq y}\langle q_l \rangle$$

$$\dfrac{\dfrac{\dfrac{\dfrac{x \doteqdot^{[]}_u y \qquad x \simeq x'}{x' \doteqdot^{[]}_u y}\langle k_a\rangle}{y \doteqdot^{[]}_u x'}\langle s_a\rangle \qquad y \simeq y'}{y' \doteqdot^{[]}_u x'}\langle k_a\rangle}{x' \doteqdot^{[]}_u y'}\langle s_a\rangle$$

□

Corollary 4.7. Let $\mathcal{C}$ be a set of constraints and $u \in A$ be an agent. We have $x \in \mathcal{D}_r(\overline{\mathcal{C}})$ if and only if $x \simeq x \in \overline{\mathcal{C}}$ if and only if $x \doteqdot^{[]}_u x \in \overline{\mathcal{C}}$.

Proof. We suppose that $x \in \mathcal{D}_r(\overline{\mathcal{C}})$. By Definition 4.3, we have $x \in \bigcup_{y \simeq z \in \overline{\mathcal{C}}}(\mathcal{E}(y) \cup \mathcal{E}(z))$ or $x \in \bigcup_{y \doteqdot^{\mu}_u z \in \overline{\mathcal{C}}}(\mathcal{E}(y) \cup \mathcal{E}(z))$. There are two cases.
- There exists $y \simeq z \in \overline{\mathcal{C}}$ such that $x \in \mathcal{E}(y) \cup \mathcal{E}(z)$. Then, there exists a resource label k such that $xk \simeq z \in \overline{\mathcal{C}}$ or $y \simeq xk \in \overline{\mathcal{C}}$. Thus, by Proposition 4.6, $x \simeq x \in \overline{\mathcal{C}}$.
- There exists $y \doteqdot^{\mu}_u z \in \overline{\mathcal{C}}$ such that $x \in \mathcal{E}(y) \cup \mathcal{E}(z)$. Then, there exists a resource label k such that $xk \doteqdot^{\mu}_u z \in \overline{\mathcal{C}}$ or $y \doteqdot^{\mu}_u xk \in \overline{\mathcal{C}}$. By the rule $\langle p_a \rangle$, $xk \doteqdot^{[]}_u z \in \overline{\mathcal{C}}$ or $y \doteqdot^{[]}_u xk \in \overline{\mathcal{C}}$. Thus, by Proposition 4.6, $x \simeq x \in \overline{\mathcal{C}}$.

We suppose that $x \simeq x \in \overline{\mathcal{C}}$. Then, by definition, $x \in \mathcal{D}_r(\overline{\mathcal{C}})$. Therefore we have $x \in \mathcal{D}_r(\overline{\mathcal{C}})$ iff $x \simeq x \in \overline{\mathcal{C}}$. Now, we have $x \simeq x \in \overline{\mathcal{C}}$if and only if $x \doteqdot^{[]}_u x \in \overline{\mathcal{C}}$ by the rules $\langle k_r \rangle$ and $\langle r_a \rangle$. □

Corollary 4.8. Let $\mathcal{C}$ be a set of constraints. If $xy \in \mathcal{D}_r(\overline{\mathcal{C}})$, $x' \simeq x \in \overline{\mathcal{C}}$ and $y' \simeq y \in \overline{\mathcal{C}}$ then $xy \simeq x'y' \in \overline{\mathcal{C}}$.

Proof. By Corollary 4.7, $xy \simeq xy \in \overline{\mathcal{C}}$. We give the following deduction tree

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{y' \simeq y} \qquad \dfrac{\dfrac{\dfrac{\vdots}{x' \simeq x} \quad \dfrac{\vdots}{xy \simeq xy}}{x'y \simeq xy}\langle c_r\rangle}{x'y \simeq x'y}\langle p_l\rangle}{x'y' \simeq x'y}\langle c_r\rangle \qquad \dfrac{\dfrac{\vdots}{x' \simeq x} \quad \dfrac{\vdots}{xy \simeq xy}}{x'y \simeq xy}\langle c_r\rangle}{x'y' \simeq xy}\langle t_r\rangle}{xy \simeq x'y'}\langle s_r\rangle$$

□

Proposition 4.9. Let $\mathcal{C}$ be a set of constraints and $u \in A$ be an agent. We have $\mu \in \mathcal{D}_a(\overline{\mathcal{C}})$ if and only if $\triangleright\mu \in \overline{\mathcal{C}}$.

Proof. By definition and by the rule $\langle a_n \rangle$. □

Proposition 4.10. Let $\mathcal{C}$ be a set of constraints. We have $\mathcal{A}_r(\mathcal{C}) = \mathcal{A}_r(\overline{\mathcal{C}})$.

Proof. As $\mathcal{C} \subseteq \overline{\mathcal{C}}$, then we have $\mathcal{A}_r(\mathcal{C}) \subseteq \mathcal{A}_r(\overline{\mathcal{C}})$. For the converse, it suffices to observe that the rules of Fig. 6 do not introduce new resource constants. Thus $\mathcal{A}_r(\overline{\mathcal{C}}) \subseteq \mathcal{A}_r(\mathcal{C})$. Therefore $\mathcal{A}_r(\mathcal{C}) = \mathcal{A}_r(\overline{\mathcal{C}})$. □

Lemma 4.11 (Compactness). Let $\mathcal{C}$ be a (possibly infinite) set of constraints:

1 If $x \simeq y \in \overline{\mathcal{C}}$ then there is a finite set $\mathcal{C}_f$ such that $\mathcal{C}_f \subseteq \mathcal{C}$ and $x \simeq y \in \overline{\mathcal{C}_f}$
2 If $x \dashrightarrow^{\mu}_{u} y \in \overline{\mathcal{C}}$ then there is a finite set $\mathcal{C}_f$ such that $\mathcal{C}_f \subseteq \mathcal{C}$ and $x \dashrightarrow^{\mu}_{u} y \in \overline{\mathcal{C}_f}$
3 If $\triangleright\mu \in \overline{\mathcal{C}}$ then there is a finite set $\mathcal{C}_f$ such that $\mathcal{C}_f \subseteq \mathcal{C}$ and $\triangleright\mu \in \overline{\mathcal{C}_f}$

Proof. Let $\mathcal{C}$ be a set of constraints. Let $c \in \overline{\mathcal{C}}$ be a constraint. If $c \in \overline{\mathcal{C}}$ because $c \in \mathcal{C}$ then by considering $\mathcal{C}_f = \{c\}$, we have $\mathcal{C}_f \subseteq \mathcal{C}$ and $c \in \overline{\mathcal{C}_f}$. In the other cases, the constraint c is obtained by rules of Fig. 6. The proof is by induction on the size n of the deduction tree of c. The detailed proof is given in Appendix A. □

4.2. *A tableaux calculus for PASL*

Now, we can define a labelled tableaux calculus for ESL in the spirit of previous works for BI [Galmiche et al. 2005] and BBI [Larchey-Wendling 2016]. Our calculus is based on some ideas and techniques coming from tableaux for Public Announcement Logic [Balbiani et al. 2010], that consists in taking into account the lists of announcements.

Definition 4.12. A *labelled formula* is a 4-tuple of the form $(\mathbb{S}\varphi : \mu, x)$, such that $\mathbb{S} \in \{\mathbb{T}, \mathbb{F}\}$, μ is a (possibly empty) list of PASL formulae, $\varphi \in \mathcal{L}$ is a formula and $x \in L_r$ is a resource label. A *constrained set of statements* (CSS) is a pair $\langle \mathcal{F}, \mathcal{C} \rangle$, where $\mathcal{F}$ is a set of labelled formulae and $\mathcal{C}$ is a set of constraints, satisfying the property:

$$\text{if } (\mathbb{S}\varphi : \mu, x) \in \mathcal{F} \text{ then } x \simeq x \in \overline{\mathcal{C}} \text{ and } \mu \in \mathcal{D}_a(\overline{\mathcal{C}}) \quad (P_{css})$$

A CSS $\langle \mathcal{F}, \mathcal{C} \rangle$ is *finite* if $\mathcal{F}$ and $\mathcal{C}$ are finite.
The relation $\preccurlyeq$ is defined by $\langle \mathcal{F}, \mathcal{C} \rangle \preccurlyeq \langle \mathcal{F}', \mathcal{C}' \rangle$ iff $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{C} \subseteq \mathcal{C}'$. We denote $\langle \mathcal{F}_f, \mathcal{C}_f \rangle \preccurlyeq_f \langle \mathcal{F}, \mathcal{C} \rangle$ when $\langle \mathcal{F}_f, \mathcal{C}_f \rangle \preccurlyeq \langle \mathcal{F}, \mathcal{C} \rangle$ holds and $\langle \mathcal{F}_f, \mathcal{C}_f \rangle$ is finite, meaning that $\mathcal{F}_f$ and $\mathcal{C}_f$ are both finite.

Proposition 4.13. For any CSS $\langle \mathcal{F}_f, \mathcal{C} \rangle$ where $\mathcal{F}_f$ is finite, there exists $\mathcal{C}_f \subseteq \mathcal{C}$ such that $\mathcal{C}_f$ is finite and $\langle \mathcal{F}_f, \mathcal{C}_f \rangle$ is a CSS.

Proof. By induction on the number of labelled formulae that belongs to $\mathcal{F}_f$ and using Proposition 4.9 and Lemma 4.11. □

Definition 4.14 (Tableau). Let $\langle \mathcal{F}_0, \mathcal{C}_0 \rangle$ be a finite CSS. A *tableau* for $\langle \mathcal{F}_0, \mathcal{C}_0 \rangle$ is a list of CSS, called *branches*, inductively built according the following rules:

1 The one branch list $[\langle \mathcal{F}_0, \mathcal{C}_0 \rangle]$ is a tableau for $\langle \mathcal{F}_0, \mathcal{C}_0 \rangle$
2 If the list $\mathcal{T}_m \oplus [\langle \mathcal{F}, \mathcal{C} \rangle] \oplus \mathcal{T}_n$ is a tableau for $\langle \mathcal{F}_0, \mathcal{C}_0 \rangle$ and

$$\frac{\text{cond}\langle \mathcal{F}, \mathcal{C} \rangle}{\langle \mathcal{F}_1, \mathcal{C}_1 \rangle \mid \ldots \mid \langle \mathcal{F}_k, \mathcal{C}_k \rangle}$$

is an instance of a rule of Fig. 7 for which the condition $\text{cond}\langle \mathcal{F}, \mathcal{C} \rangle$ is fulfilled, then the list $\mathcal{T}_m \oplus [\langle \mathcal{F} \cup \mathcal{F}_1, \mathcal{C} \cup \mathcal{C}_1 \rangle; \ldots; \langle \mathcal{F} \cup \mathcal{F}_k, \mathcal{C} \cup \mathcal{C}_k \rangle] \oplus \mathcal{T}_n$ is a tableau for $\langle \mathcal{F}_0, \mathcal{C}_0 \rangle$.

A *tableau* for the formula φ is a *tableau* for $\langle \{(\mathbb{F}\varphi : [\![]\!], c_1)\}, \{c_1 \simeq c_1\} \rangle$.

$$\frac{(\mathbb{T}\mathrm{I} : \mu, x) \in \mathcal{F}}{\langle \emptyset, \{x \simeq \varepsilon\}\rangle}\ \langle \mathbb{T}\mathrm{I}\rangle$$

$$\frac{(\mathbb{T}\neg\varphi : \mu, x) \in \mathcal{F}}{\langle \{(\mathbb{F}\varphi : \mu, x)\}, \emptyset\rangle}\ \langle \mathbb{T}\neg\rangle \qquad \frac{(\mathbb{F}\neg\varphi : \mu, x) \in \mathcal{F}}{\langle \{(\mathbb{T}\varphi : \mu, x)\}, \emptyset\rangle}\ \langle \mathbb{F}\neg\rangle$$

$$\frac{(\mathbb{T}\varphi \wedge \psi : \mu, x) \in \mathcal{F}}{\langle \{(\mathbb{T}\varphi : \mu, x), (\mathbb{T}\psi : \mu, x)\}, \emptyset\rangle}\ \langle \mathbb{T}\wedge\rangle \qquad \frac{(\mathbb{F}\varphi \wedge \psi : \mu, x) \in \mathcal{F}}{\langle \{(\mathbb{F}\varphi : \mu, x)\}, \emptyset\rangle \ \mid\ \langle \{(\mathbb{F}\psi : \mu, x)\}, \emptyset\rangle}\ \langle \mathbb{F}\wedge\rangle$$

$$\frac{(\mathbb{T}\varphi \vee \psi : \mu, x) \in \mathcal{F}}{\langle \{(\mathbb{T}\varphi : \mu, x)\}, \emptyset\rangle \ \mid\ \langle \{(\mathbb{T}\psi : \mu, x)\}, \emptyset\rangle}\ \langle \mathbb{T}\vee\rangle \qquad \frac{(\mathbb{F}\varphi \vee \psi : \mu, x) \in \mathcal{F}}{\langle \{(\mathbb{F}\varphi : \mu, x), (\mathbb{F}\psi : \mu, x)\}, \emptyset\rangle}\ \langle \mathbb{F}\vee\rangle$$

$$\frac{(\mathbb{T}\varphi \to \psi : \mu, x) \in \mathcal{F}}{\langle \{(\mathbb{F}\varphi : \mu, x)\}, \emptyset\rangle \ \mid\ \langle \{(\mathbb{T}\psi : \mu, x)\}, \emptyset\rangle}\ \langle \mathbb{T}\to\rangle \qquad \frac{(\mathbb{F}\varphi \to \psi : \mu, x) \in \mathcal{F}}{\langle \{(\mathbb{T}\varphi : \mu, x), (\mathbb{F}\psi : \mu, x)\}, \emptyset\rangle}\ \langle \mathbb{F}\to\rangle$$

$$\frac{(\mathbb{T}\varphi * \psi : \mu, x) \in \mathcal{F}}{\langle \{(\mathbb{T}\varphi : \mu, c_i), (\mathbb{T}\psi : \mu, c_j)\}, \{x \simeq c_i c_j\}\rangle}\ \langle \mathbb{T}*\rangle \qquad \frac{(\mathbb{F}\varphi * \psi : \mu, x) \in \mathcal{F} \text{ and } x \simeq yz \in \overline{\mathcal{C}}}{\langle \{(\mathbb{F}\varphi : \mu, y)\}, \emptyset\rangle \ \mid\ \langle \{(\mathbb{F}\psi : \mu, z)\}, \emptyset\rangle}\ \langle \mathbb{F}*\rangle$$

$$\frac{(\mathbb{T}\varphi \mathrel{-\!\!*} \psi : \mu, x) \in \mathcal{F} \text{ and } xy \simeq xy \in \overline{\mathcal{C}}}{\langle \{(\mathbb{F}\varphi : \mu, y)\}, \emptyset\rangle \ \mid\ \langle \{(\mathbb{T}\psi : \mu, xy)\}, \emptyset\rangle}\ \langle \mathbb{T}\mathrel{-\!\!*}\rangle \qquad \frac{(\mathbb{F}\varphi \mathrel{-\!\!*} \psi : \mu, x) \in \mathcal{F}}{\langle \{(\mathbb{T}\varphi : \mu, c_i), (\mathbb{F}\psi : \mu, xc_i)\}, \{xc_i \simeq xc_i\}\rangle}\ \langle \mathbb{F}\mathrel{-\!\!*}\rangle$$

$$\frac{(\mathbb{T}K_u\varphi : \mu, x) \in \mathcal{F} \text{ and } x =^{\mu}_{u} y \in \overline{\mathcal{C}}}{\langle \{(\mathbb{T}\varphi : \mu, y)\}, \emptyset\rangle}\ \langle \mathbb{T}K\rangle \qquad \frac{(\mathbb{F}K_u\varphi : \mu, x) \in \mathcal{F}}{\langle \{(\mathbb{F}\varphi : \mu, c_i)\}, \{x =^{\mu}_{u} c_i\}\rangle}\ \langle \mathbb{F}K\rangle$$

$$\frac{(\mathbb{T}\widetilde{K}_u\varphi : \mu, x) \in \mathcal{F}}{\langle \{(\mathbb{T}\varphi : \mu, c_i)\}, \{x =^{\mu}_{u} c_i\}\rangle}\ \langle \mathbb{T}\widetilde{K}\rangle \qquad \frac{(\mathbb{F}\widetilde{K}_u\varphi : \mu, x) \in \mathcal{F} \text{ and } x =^{\mu}_{u} y \in \overline{\mathcal{C}}}{\langle \{(\mathbb{F}\varphi : \mu, y)\}, \emptyset\rangle}\ \langle \mathbb{F}\widetilde{K}\rangle$$

$$\frac{(\mathbb{T}[\varphi]\psi : \mu, x) \in \mathcal{F}}{\langle \{(\mathbb{F}\varphi : \mu, x)\}, \emptyset\rangle \ \mid\ \langle \{(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!], x)\}, \{\rhd \mu \oplus [\![\varphi]\!]\}\rangle}\ \langle \mathbb{T}[\cdot]\rangle \qquad \frac{(\mathbb{F}[\varphi]\psi : \mu, x) \in \mathcal{F}}{\langle \{(\mathbb{T}\varphi : \mu, x), (\mathbb{F}\psi : \mu \oplus [\![\varphi]\!], x)\}, \{\rhd \mu \oplus [\![\varphi]\!]\}\rangle}\ \langle \mathbb{F}[\cdot]\rangle$$

$$\frac{(\mathbb{T}\langle\varphi\rangle\psi : \mu, x) \in \mathcal{F}}{\langle \{(\mathbb{T}\varphi : \mu, x), (\mathbb{T}\psi : \mu \oplus [\![\varphi]\!], x)\}, \{\rhd \mu \oplus [\![\varphi]\!]\}\rangle}\ \langle \mathbb{T}\langle\cdot\rangle\rangle \qquad \frac{(\mathbb{F}\langle\varphi\rangle\psi : \mu, x) \in \mathcal{F}}{\langle \{(\mathbb{F}\varphi : \mu, x)\}, \emptyset\rangle \ \mid\ \langle \{(\mathbb{F}\psi : \mu \oplus [\![\varphi]\!], x)\}, \{\rhd \mu \oplus [\![\varphi]\!]\}\rangle}\ \langle \mathbb{F}\langle\cdot\rangle\rangle$$

$$\frac{x =^{[\![\psi_1;\ldots;\psi_k]\!]}_{u} y \in \overline{\mathcal{C}}}{\langle \{(\mathbb{T}\psi_k : [\![\psi_1;\ldots;\psi_{k-1}]\!], x), (\mathbb{T}\psi_k : [\![\psi_1;\ldots;\psi_{k-1}]\!], y)\}, \emptyset\rangle \ \mid\ \langle \{(\mathbb{F}\psi_k : [\![\psi_1;\ldots;\psi_{k-1}]\!], x), (\mathbb{F}\psi_k : [\![\psi_1;\ldots;\psi_{k-1}]\!], y)\}, \emptyset\rangle}\ \langle R_{pop}\rangle$$

$$\frac{x =^{\mu}_{u} y \in \overline{\mathcal{C}} \text{ and } \rhd \mu \oplus [\![\varphi]\!] \in \overline{\mathcal{C}}}{\langle \emptyset, \{x =^{\mu\oplus[\![\varphi]\!]}_{u} y\}\rangle \ \mid\ \langle \{(\mathbb{T}\varphi : \mu, x), (\mathbb{F}\varphi : \mu, y)\}, \emptyset\rangle \ \mid\ \langle \{(\mathbb{F}\varphi : \mu, x), (\mathbb{T}\varphi : \mu, y)\}, \emptyset\rangle}\ \langle R_{push}\rangle$$

c_i and c_j are new label constants.

Fig. 7. Rules of tableaux calculus for PASL

We encode tableaux as lists of CSS, denoted $\mathcal{T}_i$, with $\oplus$ being the concatenation of lists. Then $[e_3; e_1] \oplus [e_1; e_2; e_5] = [e_3; e_1; e_1; e_2; e_5]$. Fig. 7 presents the rules of tableaux calculus for PASL. Let us note that "c_i and c_j are new label constants" means $c_i \neq c_j \in \gamma_r \setminus \mathcal{A}_r(\mathcal{C})$.

In addition we remark that a tableau for a formula φ verifies the property (P_{css}) of Definition 4.12 (by the rule $\langle r_a \rangle$) and any application of a rule of Fig. 7 provides also a tableau that verifies the property (P_{css}). An original key point is that we have a generation of formulas from constraints (see rules $\langle R_{pop} \rangle$ and $\langle R_{push} \rangle$) in addition to the generation of constraints from formulas like in previous works on BBI variants.

Definition 4.15 (Closure condition). A CSS $\langle \mathcal{F}, \mathcal{C} \rangle$ is *closed* if one of the following conditions holds, where $p \in \mathsf{Prop}$, $\varphi \in \mathcal{L}$ and μ and κ are lists of PASL formulae:

1 $(\mathbb{T}p : \mu, x) \in \mathcal{F}$, $(\mathbb{F}p : \kappa, y) \in \mathcal{F}$ and $x \simeq y \in \overline{\mathcal{C}}$,
2 $(\mathbb{T}\varphi : \mu, x) \in \mathcal{F}$, $(\mathbb{F}\varphi : \mu, y) \in \mathcal{F}$ and $x \simeq y \in \overline{\mathcal{C}}$,
3 $(\mathbb{F}\mathrm{I} : \mu, x) \in \mathcal{F}$ and $x \simeq \varepsilon \in \overline{\mathcal{C}}$,
4 $(\mathbb{F}\top : \mu, x) \in \mathcal{F}$ and
5 $(\mathbb{T}\bot : \mu, x) \in \mathcal{F}$.

A CSS is *open* if it is not closed. A tableau for φ is *closed* if all its branches are closed and a *tableau proof* for φ is a closed tableau for φ.

Let us illustrate the PASL tableau construction with the formula $\varphi \equiv [\mathrm{I} \wedge P \wedge Q](K_a P * Q)$. We first initialize a tableau for φ with $[\langle \{(\mathbb{F}\varphi : [\![]\!], c_1)\}, \{c_1 \simeq c_1\} \rangle]$ and introduce the following representation:

$[\mathcal{F}]$	$[\mathcal{C}]$
$(\mathbb{F}[\mathrm{I} \wedge P \wedge Q](K_a P * Q) : [\![]\!], c_1)$	$c_1 \simeq c_1$

The column on left-hand side represents the labelled formula sets of the CSS of the tableau ($[\mathcal{F}]$) and the column on right-hand side represents the constraint sets of the CSS of ($[\mathcal{C}]$). By applying rules on this tableau, we obtain the tableau for φ that is given in Fig. 8.
Let us note that we decorate a labelled formula (resp. constraint) with $\surd_i$ to show that we apply a rule on this formula (resp. constraint) at step i. Let us focus on rule applications at steps 6 and 5. The step 6 consists in applying the rule $\langle \mathbb{F}K \rangle$ on the labelled formula $(\mathbb{F}K_a P : [\![\mathrm{I} \wedge P \wedge Q]\!], c_1)$. Then in order to apply this rule we have to choose a new resource constant (c_2). Then we can apply the rule introducing, in the branch, the labelled formula $(\mathbb{F}P : [\![\mathrm{I} \wedge P \wedge Q]\!], c_2)$ and the agent constraint $c_1 \sim_a^{[\![\mathrm{I} \wedge P \wedge Q]\!]} c_2$. The step 5 consists in applying the rule $\langle \mathbb{F}* \rangle$ on the labelled formula $(\mathbb{F}K_a P * Q : [\![\mathrm{I} \wedge P \wedge Q]\!], c_1)$. Then we have to choose y and z such that $c_1 \simeq yz \in \overline{\mathcal{C}}$. We have $c_1 \simeq c_1 c_1 \in \overline{\mathcal{C}}$, indeed

$$\cfrac{\cfrac{\cfrac{c_1 \simeq \varepsilon \qquad \cfrac{c_1 \simeq c_1}{\varepsilon c_1 \simeq \varepsilon c_1}\ \text{(definition)}}{c_1 c_1 \simeq \varepsilon c_1}\ \langle c_r \rangle}{c_1 c_1 \simeq c_1}\ \text{(definition)}}{c_1 \simeq c_1 c_1}\ \langle s_r \rangle$$

Therefore we can choose $y = c_1$ and $z = c_1$ and apply the rule, adding to the first branch $(\mathbb{F}K_a P : [\![\mathrm{I} \wedge P \wedge Q]\!], c_1)$ and to the second branch $(\mathbb{F}Q : [\![\mathrm{I} \wedge P \wedge Q]\!], c_1)$.
We observe that at step 7, the rule $\langle R_{pop} \rangle$ is applied on the agent constraint $c_1 \sim_a^{[\![\mathrm{I} \wedge P \wedge Q]\!]} c_2$ and that the tableau branches are closed (denoted $\times$). In particular, the fifth branch on the right-hand side is closed because $(\mathbb{T}Q : [\![]\!], c_1) \in \mathcal{F}$, $(\mathbb{F}Q : [\![\mathrm{I} \wedge P \wedge Q]\!], c_1) \in \mathcal{F}$ and $c_1 \simeq c_1 \in \overline{\mathcal{C}}$. In conclusion, we have a closed tableau proof for the formula $[\mathrm{I} \wedge P \wedge Q](K_a P * Q)$.

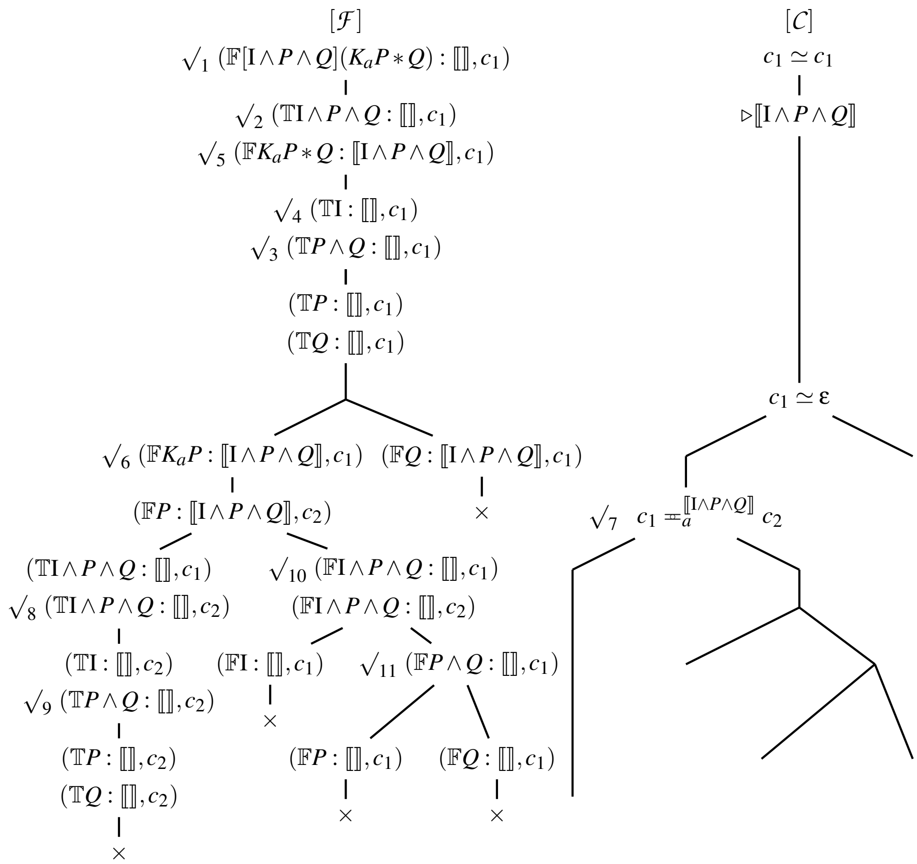


Fig. 8. Tableau for $[\mathrm{I} \wedge P \wedge Q](K_a P * Q)$

4.3. *Soundness of the PASL calculus*

In this subsection, we show the soundness of our tableaux method for PASL. The proof uses similar techniques as the ones used in BI for a labelled tableaux method [Galmiche et al. 2005]. The key point consists in considering the notion of *realizability* of a CSS $\langle \mathcal{F}, \mathcal{C} \rangle$, meaning that there exists a model $\mathcal{M}$ and an embedding $|.|$ from the resource labels to the resource set of $\mathcal{M}$ such that if $(\mathbb{T}\varphi : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$ then $|x| \vDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi$, and if $(\mathbb{F}\varphi : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$ then $|x| \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi$.
Such embeddings are firstly defined as functions $|.| : \mathcal{A}_r(\mathcal{C}) \to R$. Then, we implicitly extend them to $\mathcal{D}_r(\mathcal{C}) \rightharpoonup R$, that is for all $c_{i_1} \dots c_{i_n} \in \mathcal{D}_r(\mathcal{C})$, $|c_{i_1} \dots c_{i_n}| = |c_{i_1}| \bullet \dots \bullet |c_{i_n}|$ and $|\varepsilon| = e$. We remark that $|x|$ can be undefined, because resource composition is a partial function.

Definition 4.16 (Realization). Let $\langle \mathcal{F}, \mathcal{C} \rangle$ be a CSS. A *realization* of $\langle \mathcal{F}, \mathcal{C} \rangle$ is a pair $\mathfrak{R} = (\mathcal{M}, |.|)$ where $\mathcal{M} = (\mathcal{R}, \{\sim_a\}_{a \in A}, V)$ is a model and $|.| : \mathcal{D}_r(\mathcal{C}) \to R$, such that:

— $|\varepsilon| = e$
— $|.|$ is a total function: for all $x \in \mathcal{D}_r(\mathcal{C})$, $|x|$ is defined

— If $(\mathbb{T}\varphi : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$ then $|x| \models_{\mathcal{M}|\psi_1|...|\psi_k} \varphi$
— If $(\mathbb{F}\varphi : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$ then $|x| \not\models_{\mathcal{M}|\psi_1|...|\psi_k} \varphi$
— If $x \simeq y \in \mathcal{C}$ then $|x| = |y|$
— If $x =_u^{[\![\psi_1;...;\psi_k]\!]} y \in \mathcal{C}$ then $|x| \sim_u |y|$ in the updated model $\mathcal{M}|\psi_1|...|\psi_k$

We say that a CSS is *realizable* if there exists a realization of this CSS. We say that a tableau is *realizable* if at least one of its branches is realizable.

Proposition 4.17. Let $\langle \mathcal{F}, \mathcal{C} \rangle$ be a CSS and $\mathfrak{R} = (\mathcal{M}, |.|)$ a realization of it. $\mathfrak{R}$ is a realization of $\langle \mathcal{F}, \overline{\mathcal{C}} \rangle$, in other words:

1 For all $x \in \mathcal{D}_r(\overline{\mathcal{C}})$, $|x|$ is defined
2 If $x \simeq y \in \overline{\mathcal{C}}$ then $|x| = |y|$
3 If $x =_u^{[\![\psi_1;...;\psi_k]\!]} y \in \overline{\mathcal{C}}$ then $|x| \sim_u |y|$ in the updated model $\mathcal{M}|\psi_1|...|\psi_k$

Proof. The detailed proof is given in Appendix B. □

Lemma 4.18. Rules of the PASL tableaux calculus preserve realizability.

Proof. Let $\mathcal{T}$ be a realizable tableau. By definition, $\mathcal{T}$ contains a realizable branch $\mathcal{B} = \langle \mathcal{F}, \mathcal{C} \rangle$. Let $\mathfrak{R} = (\mathcal{M}, |.|)$ a realization of the branch $\mathcal{B}$, where $\mathcal{M} = (\mathcal{R}, \{\sim_a\}_{a\in A}, V)$ and $|.| : \mathcal{D}_r(\mathcal{C}) \to R$. If we apply a rule on a labelled formula of another branch than $\mathcal{B}$ then $\mathcal{B}$ is not modified and $\mathcal{T}$ stays realizable. Else, we prove by case on the formula or agent constraint whose is applied the rule. The detailed proof is given in Appendix C. □

Lemma 4.19. Closed branches are not realizable.

Proof. Let $\langle \mathcal{F}, \mathcal{C} \rangle$ a closed branch. We suppose that it is realizable. Let $\mathfrak{R} = (\mathcal{M}, |.|)$ a realization of it. Then we consider the five cases of closure in Definition 4.15. For instance, for $(\mathbb{T}p : [\![\varphi_1;...;\varphi_k]\!], x) \in \mathcal{F}$, $(\mathbb{F}p : [\![\psi_1;...;\psi_l]\!], y) \in \mathcal{F}$ and $x \simeq y \in \overline{\mathcal{C}}$, by definition of realization and Proposition 4.17, we have $|x| \models_{\mathcal{M}|\varphi_1|...|\varphi_k} p$, $|y| \not\models_{\mathcal{M}|\psi_1|...|\psi_l} p$ and $|x| = |y|$. Again by definition, we have $|x| \in V(p)$ and $|y| \notin V(p)$, which is absurd. The other cases are similar and as all cases are absurd, we conclude that $\langle \mathcal{F}, \mathcal{C} \rangle$ is not realizable. □

Theorem 4.20 (Soundness). If there exists a tableau proof for a formula φ then φ is valid.

Proof. We suppose that there exists a tableau proof for φ. Then there is a closed tableau $\mathcal{T}_\varphi$ for the CSS $\mathfrak{C} = \langle \{(\mathbb{F}\varphi : [\![]\!], c_1)\}, \{c_1 \simeq c_1\} \rangle$. Let us now suppose that φ is not valid. Then there is a countermodel $\mathcal{M} = (\mathcal{R}, \{\sim_a\}_{a\in A}, V)$ and a resource $r \in R$ such that $r \not\models_{\mathcal{M}} \varphi$. Let $\mathfrak{R} = (\mathcal{M}, |.|)$ such that $|c_1| = r$. We remark that $\mathfrak{R}$ is a realization of $\mathfrak{C}$. By Lemma 4.18, $\mathcal{T}_\varphi$ is realizable. By Lemma 4.19, $\mathcal{T}_\varphi$ cannot be closed. But, this is absurd because $\mathcal{T}_\varphi$ is a tableau proof and is by definition closed. Therefore φ is valid. □

Coming back to our example, as we have a closed tableau proof for $[\mathrm{I} \wedge P \wedge Q](K_a P * Q)$ we can deduce that this formula is valid.

4.4. *A Countermodel Extraction Method*

We propose a countermodel extraction method, adapted from [Larchey-Wendling 2016], that consists in transforming the sets of constraints of a branch $\langle \mathcal{F}, \mathcal{C} \rangle$ into a model $\mathcal{M}$ such that if $(\mathbb{T}\varphi : x, \in)\mathcal{F}$ then $[x] \vDash_{\mathcal{M}} \varphi$ and if $(\mathbb{F}\varphi : x, \in)\mathcal{F}$ then $[x] \nvDash_{\mathcal{M}} \varphi$, where $[x]$ is the equivalence class of x, First we have to define when a CSS $\langle \mathcal{F}, \mathcal{C} \rangle$ is a *Hintikka CSS*.

Definition 4.21 (Hintikka CSS). A CSS $\langle \mathcal{F}, \mathcal{C} \rangle$ is a *Hintikka CSS* iff for any formula $\varphi, \psi, \psi_1, .., \psi_k \in \mathcal{L}$ and any resource label $x, y \in L_r$, any agent $u \in A$ and any list of formulae μ and κ:

1 $(\mathbb{T}p : \mu, x) \notin \mathcal{F}$ or $(\mathbb{F}p : \kappa, y) \notin \mathcal{F}$ or $x \simeq y \notin \overline{\mathcal{C}}$
2 $(\mathbb{T}\varphi : \mu, x) \notin \mathcal{F}$ or $(\mathbb{F}\varphi : \mu, y) \notin \mathcal{F}$ or $x \simeq y \notin \overline{\mathcal{C}}$
3 $(\mathbb{F}\mathrm{I} : \mu, x) \notin \mathcal{F}$ or $x \simeq \varepsilon \notin \overline{\mathcal{C}}$
4 $(\mathbb{F}\top : \mu, x) \notin \mathcal{F}$
5 $(\mathbb{T}\bot : \mu, x) \notin \mathcal{F}$
6 If $(\mathbb{T}\mathrm{I} : \mu, x) \in \mathcal{F}$ then $x \simeq \varepsilon \in \overline{\mathcal{C}}$
7 If $(\mathbb{T}\neg\varphi : \mu, x) \in \mathcal{F}$ then $(\mathbb{F}\varphi : \mu, x) \in \mathcal{F}$
8 If $(\mathbb{F}\neg\varphi : \mu, x) \in \mathcal{F}$ then $(\mathbb{T}\varphi : \mu, x) \in \mathcal{F}$
9 If $(\mathbb{T}\varphi \wedge \psi : \mu, x) \in \mathcal{F}$ then $(\mathbb{T}\varphi : \mu, x) \in \mathcal{F}$ and $(\mathbb{T}\psi : \mu, x) \in \mathcal{F}$
10 If $(\mathbb{F}\varphi \wedge \psi : \mu, x) \in \mathcal{F}$ then $(\mathbb{F}\varphi : \mu, x) \in \mathcal{F}$ or $(\mathbb{F}\psi : \mu, x) \in \mathcal{F}$
11 If $(\mathbb{T}\varphi \vee \psi : \mu, x) \in \mathcal{F}$ then $(\mathbb{T}\varphi : \mu, x) \in \mathcal{F}$ or $(\mathbb{T}\psi : \mu, x) \in \mathcal{F}$
12 If $(\mathbb{F}\varphi \vee \psi : \mu, x) \in \mathcal{F}$ then $(\mathbb{F}\varphi : \mu, x) \in \mathcal{F}$ and $(\mathbb{F}\psi : \mu, x) \in \mathcal{F}$
13 If $(\mathbb{T}\varphi \rightarrow \psi : \mu, x) \in \mathcal{F}$ then $(\mathbb{F}\varphi : \mu, x) \in \mathcal{F}$ or $(\mathbb{T}\psi : \mu, x) \in \mathcal{F}$
14 If $(\mathbb{F}\varphi \rightarrow \psi : \mu, x) \in \mathcal{F}$ then $(\mathbb{T}\varphi : \mu, x) \in \mathcal{F}$ and $(\mathbb{F}\psi : \mu, x) \in \mathcal{F}$
15 If $(\mathbb{T}\varphi * \psi : \mu, x) \in \mathcal{F}$ then $\exists y, z \in L_r, x \simeq yz \in \overline{\mathcal{C}}$ and $(\mathbb{T}\varphi : \mu, y) \in \mathcal{F}$ and $(\mathbb{T}\psi : \mu, z) \in \mathcal{F}$
16 If $(\mathbb{F}\varphi * \psi : \mu, x) \in \mathcal{F}$ then $\forall y, z \in L_r, x \simeq yz \in \overline{\mathcal{C}} \Rightarrow (\mathbb{F}\varphi : \mu, y) \in \mathcal{F}$ or $(\mathbb{F}\psi : \mu, z) \in \mathcal{F}$
17 If $(\mathbb{T}\varphi \mathrel{-\!\!*} \psi : \mu, x) \in \mathcal{F}$ then $\forall y \in L_r, xy \in \mathcal{D}_r(\overline{\mathcal{C}}) \Rightarrow (\mathbb{F}\varphi : \mu, y) \in \mathcal{F}$ or $(\mathbb{T}\psi : \mu, xy) \in \mathcal{F}$
18 If $(\mathbb{F}\varphi \mathrel{-\!\!*} \psi : \mu, x) \in \mathcal{F}$ then $\exists y \in L_r, xy \in \mathcal{D}_r(\overline{\mathcal{C}})$ and $(\mathbb{T}\varphi : \mu, y) \in \mathcal{F}$ and $(\mathbb{F}\psi : \mu, xy) \in \mathcal{F}$
19 If $(\mathbb{T}K_u\varphi : \mu, x) \in \mathcal{F}$ then $\forall y \in L_r, x \eqcirc_u^{\mu} y \in \overline{\mathcal{C}} \Rightarrow (\mathbb{T}\varphi : \mu, y) \in \mathcal{F}$
20 If $(\mathbb{F}K_u\varphi : \mu, x) \in \mathcal{F}$ then $\exists y \in L_r, x \eqcirc_u^{\mu} y \in \overline{\mathcal{C}}$ and $(\mathbb{F}\varphi : \mu, y) \in \mathcal{F}$
21 If $(\mathbb{T}\widetilde{K}_u\varphi : \mu, x) \in \mathcal{F}$ then $\exists y \in L_r, x \eqcirc_u^{\mu} y \in \overline{\mathcal{C}}$ and $(\mathbb{T}\varphi : \mu, y) \in \mathcal{F}$
22 If $(\mathbb{F}\widetilde{K}_u\varphi : \mu, x) \in \mathcal{F}$ then $\forall y \in L_r, x \eqcirc_u^{\mu} y \in \overline{\mathcal{C}} \Rightarrow (\mathbb{F}\varphi : \mu, y) \in \mathcal{F}$
23 If $(\mathbb{T}[\varphi]\psi : \mu, x) \in \mathcal{F}$ then $(\mathbb{F}\varphi : \mu, x) \in \mathcal{F}$ or $(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!], x) \in \mathcal{F}$
24 If $(\mathbb{F}[\varphi]\psi : \mu, x) \in \mathcal{F}$ then $(\mathbb{T}\varphi : \mu, x) \in \mathcal{F}$ and $(\mathbb{F}\psi : \mu \oplus [\![\varphi]\!], x) \in \mathcal{F}$
25 If $(\mathbb{T}\langle\varphi\rangle\psi : \mu, x) \in \mathcal{F}$ then $(\mathbb{T}\varphi : \mu, x) \in \mathcal{F}$ and $(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!], x) \in \mathcal{F}$
26 If $(\mathbb{F}\langle\varphi\rangle\psi : \mu, x) \in \mathcal{F}$ then $(\mathbb{F}\varphi : \mu, x) \in \mathcal{F}$ or $(\mathbb{F}\psi : \mu \oplus [\![\varphi]\!], x) \in \mathcal{F}$
27 If $x \eqcirc_u^{[\![\psi_1;\ldots;\psi_k]\!]} y \in \overline{\mathcal{C}}$ then one at least of these conditions holds:
- $(\mathbb{T}\psi_k : [\![\psi_1;\ldots;\psi_{k-1}]\!], x) \in \mathcal{F}$ and $(\mathbb{T}\psi_k : [\![\psi_1;\ldots;\psi_{k-1}]\!], y) \in \mathcal{F}$;
- $(\mathbb{F}\psi_k : [\![\psi_1;\ldots;\psi_{k-1}]\!], x) \in \mathcal{F}$ and $(\mathbb{F}\psi_k : [\![\psi_1;\ldots;\psi_{k-1}]\!], y) \in \mathcal{F}$.

28 If $x \eqcirc_u^{\mu} y \in \overline{\mathcal{C}}$ and $\triangleright\mu \oplus [\![\varphi]\!] \in \overline{\mathcal{C}}$ then one at least of these conditions holds:
- $x \eqcirc_u^{\mu \oplus [\![\varphi]\!]} y \in \overline{\mathcal{C}}$;
- $(\mathbb{T}\varphi : \mu, x) \in \mathcal{F}$ and $(\mathbb{F}\varphi : \mu, y) \in \mathcal{F}$;
- $(\mathbb{F}\varphi : \mu, x) \in \mathcal{F}$ and $(\mathbb{T}\varphi : \mu, y) \in \mathcal{F}$.

In this definition, the four first conditions certify that a Hintikka CSS is not closed and the other that all labelled formulae of a Hintikka CSS are fulfilled [Larchey-Wendling 2016].
In order to extract a countermodel from a Hintikka CSS, we manipulate equivalence classes. The equivalence class of $x \in \mathcal{D}_r(\overline{\mathcal{C}})$ is the set $[x] = \{y \in L_r \mid x \simeq y \in \overline{\mathcal{C}}\}$. We also denote $\mathcal{D}_r(\overline{\mathcal{C}})/\simeq = \{[x] \mid x \in \mathcal{D}_r(\overline{\mathcal{C}})\}$ the set of all equivalence classes of $\mathcal{D}_r(\overline{\mathcal{C}})$. We observe that $\simeq$ is an equivalence relation, because it is reflexive (by Corollary 4.7), symmetric (by rule $\langle s_r \rangle$) and transitive (by rule $\langle t_r \rangle$). Then we define a function Ω that allows us to extract a countermodel from a Hintikka CSS.

Definition 4.22 (Function Ω). Let $\langle \mathcal{F}, \mathcal{C} \rangle$ be a Hintikka CSS. The function Ω associates to $\langle \mathcal{F}, \mathcal{C} \rangle$ a 3-uplet $\Omega(\langle \mathcal{F}, \mathcal{C} \rangle) = (\mathcal{R}, \{\sim_a\}_{a \in A}, V)$, where $\mathcal{R} = (R, \bullet, e)$, such that:

— $R = \mathcal{D}_r(\overline{\mathcal{C}})/\simeq$

— $e = [\varepsilon]$

— $[x] \bullet [y] = \begin{cases} \uparrow & \text{if } xy \notin \mathcal{D}_r(\overline{\mathcal{C}}) \\ [xy] & \text{otherwise} \end{cases}$

— For all $a \in A$, $[x] \sim_a [y]$ iff $x =_a^{[\![]\!]} y \in \overline{\mathcal{C}}$

— $[x] \in V(p)$ iff $\exists y \in L_r$ such that $y \simeq x \in \overline{\mathcal{C}}$ and $(\mathbb{T}p : \mu, y) \in \mathcal{F}$

Lemma 4.23. Let $\langle \mathcal{F}, \mathcal{C} \rangle$ be a Hintikka CSS. $\Omega(\langle \mathcal{F}, \mathcal{C} \rangle)$ is a model.

Proof. We show that $\Omega(\langle \mathcal{F}, \mathcal{C} \rangle) = (\mathcal{R}, \{\sim_a\}_{a \in A}, V)$, where $\mathcal{R} = (R, \bullet, e)$, is a model.
We first show that $\mathcal{R} = (R, \bullet, e)$ is a PRM and then that, for all $a \in A$,$\sim_a \subseteq R \times R$ is well-defined, reflexive, symmetric and transitive. Obviously, the valuation V is well-formed, meaning that if $[x] \in V(p)$ and $x \simeq x' \in \overline{\mathcal{C}}$ then $[x'] \in V(p)$. □

Lemma 4.24. Let $\langle \mathcal{F}, \mathcal{C} \rangle$ be a Hintikka CSS and $\mathcal{M} = \Omega(\langle \mathcal{F}, \mathcal{C} \rangle) = (\mathcal{R}, \{\sim_a\}_{a \in A}, V)$, where $\mathcal{R} = (R, \bullet, e)$. For all formulae $\varphi \in \mathcal{L}$, all $[\![\psi_1; ...; \psi_k]\!] \in \mathcal{D}_a(\overline{\mathcal{C}})$, all agents $a \in A$ and all $x, y \in \mathcal{D}_r(\overline{\mathcal{C}})$, we have:

(1) $x =_a^{[\![\psi_1;...;\psi_k]\!]} y \in \overline{\mathcal{C}}$ iff $[x] \sim_a [y]$ in the model $\mathcal{M}|\psi_1|...|\psi_k$
(2) If $(\mathbb{F}\varphi : [\![\psi_1; ...; \psi_k]\!], x) \in \mathcal{F}$ then $[x] \not\models_{\mathcal{M}|\psi_1|...|\psi_k} \varphi$
(3) If $(\mathbb{T}\varphi : [\![\psi_1; ...; \psi_k]\!], x) \in \mathcal{F}$ then $[x] \models_{\mathcal{M}|\psi_1|...|\psi_k} \varphi$

Proof. These properties are proved simultaneously by induction on $len(x =_a^{[\![\psi_1;...;\psi_k]\!]} y)$ and $len(\mathcal{M}|\psi_1|...|\psi_k)$ for the property (1) and on $len(\mathbb{S}\varphi : \mu, x)$ for the properties (2) and (3), where len is defined as follows :
$len(x =_a^{[\![\psi_1;...;\psi_k]\!]} y) \;=\; 2 + len([\![\psi_1; ...; \psi_k]\!])$
$len(\mathcal{M}|\psi_1|...|\psi_k) = 2 + len([\![\psi_1; ...; \psi_k]\!])$
$len(\mathbb{S}\varphi : [\![\psi_1; ...; \psi_k]\!], x) = 1 + len(\varphi) + len([\![\psi_1; ...; \psi_k]\!])$
$len(p) = len(\top) = len(\bot) = len(\mathrm{I}) = 1.$
$len(\varphi_1 \wedge \varphi_2) = len(\varphi_1 \vee \varphi_2) = len(\varphi_1 \to \varphi_2) = len(\varphi_1 * \varphi_2) = len(\varphi_1 \mathrel{-\!\!*} \varphi_2)$
$= 1 + len(\varphi_1) + len(\varphi_2)$
$len(\neg\varphi) = len(K_a\varphi) = len(\widetilde{K}_a\varphi) = 1 + len(\varphi)$
$len([\varphi_1]\varphi_2) = len(\langle\varphi_1\rangle\varphi_2) = 1 + len(\varphi_1) + len(\varphi_2)$
$len([\![\psi_1; ...; \psi_k]\!]) = len(\psi_1) + ... + len(\psi_k).$

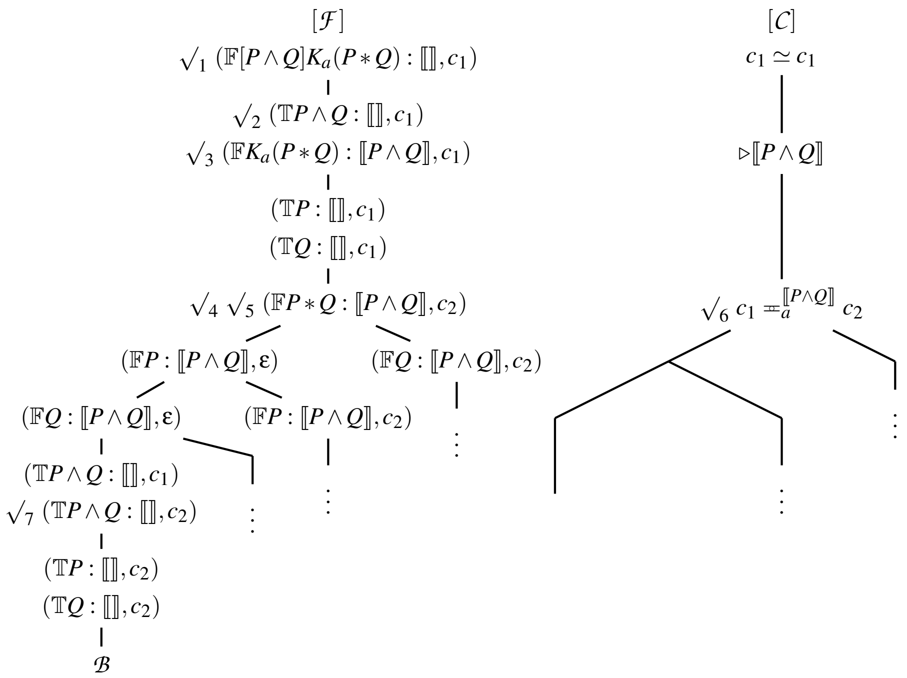


Fig. 9. Tableau for $[P \wedge Q]K_a(P * Q)$

Let us note that this induction is more complex than the one used in the proof of the corresponding result for some BBI variants [Courtault and Galmiche 2015]. The detailed proof is given in appendix D. □

Lemma 4.25. Let $\langle \mathcal{F}, \mathcal{C} \rangle$ be a Hintikka CSS such that $(\mathbb{F}\varphi : x, \in)\mathcal{F}$. The formula φ is not valid and $\Omega(\langle \mathcal{F}, \mathcal{C} \rangle)$ is a countermodel of φ.

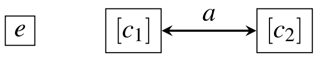
Proof. Let $\langle \mathcal{F}, \mathcal{C} \rangle$ be a Hintikka CSS such that $(\mathbb{F}\varphi : [\![]\!], x) \in \mathcal{F}$. Let $\mathcal{K} = \Omega(\langle \mathcal{F}, \mathcal{C} \rangle)$. By Lemma 4.23, $\mathcal{K}$ is a model. As $\langle \mathcal{F}, \mathcal{C} \rangle$ is a CSS, then by (P_{css}) and Corollary 4.7, $x \in \mathcal{D}_r(\overline{\mathcal{C}})$. Thus, by Lemma 4.24, we have $[x] \not\models_{\mathcal{M}} \varphi$. Therefore $\mathcal{K}$ is a countermodel of the formula φ and we can conclude that φ is not valid. □

Let us illustrate this extraction method with an example. We consider that $A = \{a\}$ and we show that the formula $[P \wedge Q]K_a(P * Q)$ is not valid and we extract a countermodel of it. By applying tableau rules, we obtain the tableau of Fig. 9.
This tableau contains a branch (denoted $\mathcal{B}$) which is a Hintikka CSS. By Lemma 4.25, we can deduce that $[P \wedge Q]K_a(P * Q)$ is not valid and $\Omega(\mathcal{B})$ is a countermodel of this formula.
We extract this countermodel, using Definition 4.22.
We have $\mathcal{M} = \Omega(\mathcal{B}) = (\mathcal{R}, \{\sim_a\}_{a \in A}, V)$, where $\mathcal{R} = (R, \bullet, e)$, such that:

— $R = \mathcal{D}_r(\overline{\mathcal{C}})/\simeq\; = \{e, [c_1], [c_2]\}$, where $e = [\varepsilon]$.
— The resource composition $\bullet$ is given by:

$\bullet$	e	$[c_1]$	$[c_2]$
e	e	$[c_1]$	$[c_2]$
$[c_1]$	$[c_1]$	$\uparrow$	$\uparrow$
$[c_2]$	$[c_2]$	$\uparrow$	$\uparrow$

— The equivalence relation, where the reflexivity is not represented:



— $V(P) = \{[c_1],[c_2]\}$ and $V(Q) = \{[c_1],[c_2]\}$

We can easily verify that this model is a countermodel of $[P \wedge Q]K_a(P * Q)$.

4.5. *Completeness of the PASL calculus*

The proof of completeness for PASL is an extension of the one for BBI [Larchey-Wendling 2016] to our epistemic connectives. It consists in building a Hintikka CSS from a CSS which can be closed. In order to build this Hintikka CSS, we use a fair strategy and an oracle.

Definition 4.26 (Fair strategy). A *fair strategy* is a sequence of labelled formulae, agent constraints and announcement constraint $(S_i)_{i\in\mathbb{N}}$ in $(\{\mathbb{T},\mathbb{F}\} \times \mathcal{L} \times \mathfrak{L}_{\mathcal{L}} \times L_r) \cup (L_r \times \mathfrak{L}_{\mathcal{L}} \times A \times L_r) \cup (\mathfrak{L}_{\mathcal{L}})$ such that all labelled formulae, all agent constraints and all announcement constraints occur infinitely many times in this sequence, that is $\{i \in \mathbb{N} \mid S_i \equiv (\mathbb{S}F : \mu, x)\}$, $\{i \in \mathbb{N} \mid S_i \equiv x \xrightarrow{\mu}_u y\}$ and $\{i \in \mathbb{N} \mid S_i \equiv \rhd\mu\}$ are infinite for any $(\mathbb{S}F : \mu, x) \in \{\mathbb{T},\mathbb{F}\} \times \mathcal{L} \times \mathfrak{L}_{\mathcal{L}} \times L_r$, any $x \xrightarrow{\mu}_u y \in L_r \times \mathfrak{L}_{\mathcal{L}} \times A \times L_r$ and any $\rhd\mu \in \mathfrak{L}_{\mathcal{L}}$.

Proposition 4.27. There exists a fair strategy.

Proof. Let $X = (\{\mathbb{T},\mathbb{F}\} \times \mathcal{L} \times \mathfrak{L}_{\mathcal{L}} \times L_r) \cup (L_r \times \mathfrak{L}_{\mathcal{L}} \times A \times L_r) \cup (\mathfrak{L}_{\mathcal{L}})$. As *Prop* is countable then $\mathcal{L}$ is countable and then $\mathfrak{L}_{\mathcal{L}}$ is also countable. Moreover, L_r is countable (remember that γ_r is countable). Therefore, X is countable. Then $\mathbb{N} \times X$ is countable and there exists a surjective function $\varphi : \mathbb{N} \longrightarrow \mathbb{N} \times X$. Let $p : \mathbb{N} \times X \longrightarrow X$ defined by $p(i,x) = x$ and $u = p \circ \varphi$. We show that u is a fair strategy by showing that for any $x \in X$, $u^{-1}(\{x\})$ is infinite. Let $x \in X$. $u^{-1}(\{x\}) = \varphi^{-1}(p^{-1}(\{x\}))$. But $p^{-1}(\{x\}) = \{(i,x) | i \in \mathbb{N}\}$ then $p^{-1}(x)$ is infinite. As φ is surjective $\varphi^{-1}(p^{-1}(\{x\}))$ is also infinite. □

Definition 4.28. Let $\mathcal{P}$ be a set of CSS.

1 $\mathcal{P}$ is $\preccurlyeq$*-closed* if $\langle \mathcal{F}, \mathcal{C} \rangle \in \mathcal{P}$ holds whenever $\langle \mathcal{F}, \mathcal{C} \rangle \preccurlyeq \langle \mathcal{F}', \mathcal{C}' \rangle$ and $\langle \mathcal{F}', \mathcal{C}' \rangle \in \mathcal{P}$ holds.

2 $\mathcal{P}$ is of *finite character* if $\langle \mathcal{F}, \mathcal{C} \rangle \in \mathcal{P}$ holds whenever $\langle \mathcal{F}_f, \mathcal{C}_f \rangle \in \mathcal{P}$ holds for $\langle \mathcal{F}_f, \mathcal{C}_f \rangle \preccurlyeq_f \langle \mathcal{F}, \mathcal{C} \rangle$.

3 $\mathcal{P}$ is *saturated* if for any $\langle \mathcal{F}, \mathcal{C} \rangle \in \mathcal{P}$ and any instance

$$\frac{cond(\mathcal{F}, \mathcal{C})}{\langle \mathcal{F}_1, \mathcal{C}_1 \rangle \mid \ldots \mid \langle \mathcal{F}_k, \mathcal{C}_k \rangle}$$

of a rule of Fig. 7, if the condition $cond(\mathcal{F}, \mathcal{C})$ is fulfilled then $\langle \mathcal{F} \cup \mathcal{F}_i, \mathcal{C} \cup \mathcal{C}_i \rangle \in \mathcal{P}$ for at least one $i \in \{1, \ldots, k\}$.

Definition 4.29 (Oracle). An *oracle* is a set of non closed CSS which is $\preccurlyeq$-closed, of finite character and saturated.

Lemma 4.30. There exists an oracle which contains every finite CSS for which there exists no closed tableau.

Proof. The proof is an adaptation of proof of completeness of tableaux for BBI [Larchey-Wendling 2016]. The detailed proof is given in Appendix G. □

In order to prove the completeness of our tableau calculus we consider a formula φ for which there exists no proof and we show that there exists a countermodel for this formula. We denote $\mathcal{T}_0$ the initial tableau for φ. Then, we have

1 $\mathcal{T}_0 = [\langle\{(\mathbb{F}\varphi : [\![]\!], c_1)\}, \{c_1 \simeq c_1\}\rangle]$
2 $\mathcal{T}_0$ cannot be closed

Now, we present a way to obtain a Hintikka CSS, which will allow us to conclude to the completeness. By Lemma 4.30, there exists an oracle which contains every finite CSS for which there exists no closed tableau. We denote $\mathcal{P}$ this oracle. By Proposition 4.27, there exists a fair strategy. We denote $\mathcal{S}$ this strategy and S_i the i^{th} formula or agent constraint of $\mathcal{S}$. As $\mathcal{T}_0$ cannot be closed then its unique branch belongs to the oracle, that is $\langle\{(\mathbb{F}\varphi : [\![]\!], c_1)\}, \{c_1 \simeq c_1\}\rangle \in \mathcal{P}$.

Now, we built a sequence $\langle \mathcal{F}_i, \mathcal{C}_i\rangle_{i \geqslant 0}$ as follows:

— $\langle \mathcal{F}_0, \mathcal{C}_0\rangle = \langle\{(\mathbb{F}\varphi : [\![]\!], c_1)\}, \{c_1 \simeq c_1\}\rangle$
— S_i is a labelled formula of the form $(\mathbb{S}F_i : \mu, x)$:
  - If $\langle \mathcal{F}_i \cup \{(\mathbb{S}F_i : \mu, x)\}, \mathcal{C}_i\rangle \notin \mathcal{P}$ then we have $\langle \mathcal{F}_{i+1}, \mathcal{C}_{i+1}\rangle = \langle \mathcal{F}_i, \mathcal{C}_i\rangle$
  - If $\langle \mathcal{F}_i \cup \{(\mathbb{S}F_i : \mu, x)\}, \mathcal{C}_i\rangle \in \mathcal{P}$ then we have $\langle \mathcal{F}_{i+1}, \mathcal{C}_{i+1}\rangle = \langle \mathcal{F}_i \cup \{(\mathbb{S}F_i : \mu, x)\} \cup F_e, \mathcal{C}_i \cup \mathcal{C}_e\rangle$ such that F_e and $\mathcal{C}_e$ are determined by:

$\mathbb{S}$	F_i	F_e	$\mathcal{C}_e$
$\mathbb{T}$	I	$\emptyset$	$\{x \simeq \varepsilon\}$
$\mathbb{T}$	$\varphi * \psi$	$\{(\mathbb{T}\varphi : \mu, \mathfrak{a}), (\mathbb{T}\psi : \mu, \mathfrak{b})\}$	$\{x \simeq \mathfrak{a}\mathfrak{b}\}$
$\mathbb{F}$	$\varphi \mathrel{-\!\!*} \psi$	$\{(\mathbb{T}\varphi : \mu, \mathfrak{a}), (\mathbb{F}\psi : \mu, x\mathfrak{a})\}$	$\{x\mathfrak{a} \simeq x\mathfrak{a}\}$
$\mathbb{F}$	$K_u\varphi$	$\{(\mathbb{F}\varphi : \mu, \mathfrak{a})\}$	$\{x \eqsim^{\mu}_{u} \mathfrak{a}\}$
$\mathbb{T}$	$\widetilde{K}_u\varphi$	$\{(\mathbb{T}\varphi : \mu, \mathfrak{a})\}$	$\{x \eqsim^{\mu}_{u} \mathfrak{a}\}$
Otherwise		$\emptyset$	$\emptyset$

with $\mathfrak{a} = c_{2i+2}$ and $\mathfrak{b} = c_{2i+3}$.

— S_i is an agent constraint of the form $x \eqsim^{\mu}_{u} y$:
  - If $\gamma_r \cap (\mathcal{E}(x) \cup \mathcal{E}(y)) \not\subseteq \{c_1, \ldots, c_{2i+1}\}$ then we have $\langle \mathcal{F}_{i+1}, \mathcal{C}_{i+1}\rangle = \langle \mathcal{F}_i, \mathcal{C}_i\rangle$
  - If $\langle \mathcal{F}_i, \mathcal{C}_i \cup \{x \eqsim^{\mu}_{u} y\}\rangle \notin \mathcal{P}$ then we have $\langle \mathcal{F}_{i+1}, \mathcal{C}_{i+1}\rangle = \langle \mathcal{F}_i, \mathcal{C}_i\rangle$
  - If $\langle \mathcal{F}_i, \mathcal{C}_i \cup \{x \eqsim^{\mu}_{u} y\}\rangle \in \mathcal{P}$ then we have $\langle \mathcal{F}_{i+1}, \mathcal{C}_{i+1}\rangle = \langle \mathcal{F}_i, \mathcal{C}_i \cup \{x \eqsim^{\mu}_{u} y\}\rangle$

Proposition 4.31. For any $i \in \mathbb{N}$, the following properties hold:

1 $(\mathbb{F}\varphi : [\![]\!], c_1) \in \mathcal{F}_i$ and $c_1 \simeq c_1 \in \mathcal{C}_i$
2 $\mathcal{F}_i \subseteq \mathcal{F}_{i+1}$ and $\mathcal{C}_i \subseteq \mathcal{C}_{i+1}$
3 $\langle \mathcal{F}_i, \mathcal{C}_i \rangle_{i \geqslant 0} \in \mathcal{P}$
4 $\mathcal{A}_r(\mathcal{C}_i) \subseteq \{c_1, c_2, \ldots, c_{2i+1}\}$

Proof. Property 1. holds for $i = 0$. As $\langle \mathcal{F}_{i+1}, \mathcal{C}_{i+1} \rangle$ is an extension ($\cup$) of $\langle \mathcal{F}_i, \mathcal{C}_i \rangle$ then this property also holds for all $i \geqslant 0$. Property 2. holds because $\langle \mathcal{F}_{i+1}, \mathcal{C}_{i+1} \rangle$ is an extension ($\cup$) of $\langle \mathcal{F}_i, \mathcal{C}_i \rangle$. Properties 3. and 4. are proved by simultaneous induction on i. The detailed proof is given in Appendix E □

We now consider the limit CSS $\langle \mathcal{F}_\infty, \mathcal{C}_\infty \rangle$ of the sequence $\langle \mathcal{F}_i, \mathcal{C}_i \rangle_{i \geqslant 0}$ defined by:

$$\mathcal{F}_\infty = \bigcup_{i \geqslant 0} \mathcal{F}_i \quad \text{and} \quad \mathcal{C}_\infty = \bigcup_{i \geqslant 0} \mathcal{C}_i$$

Proposition 4.32. The following properties hold:

1 $\langle \mathcal{F}_\infty, \mathcal{C}_\infty \rangle \in \mathcal{P}$
2 For any labelled formula $(\mathbb{S}\varphi : \mu, x)$, if $\langle \mathcal{F}_\infty \cup \{(\mathbb{S}\varphi : \mu, x)\}, \mathcal{C}_\infty \rangle \in \mathcal{P}$ then $(\mathbb{S}\varphi : \mu, x) \in \mathcal{F}_\infty$.
3 For any agent constraint $x \eqcirc_u^\mu y$, if $\langle \mathcal{F}_\infty, \mathcal{C}_\infty \cup \{x \eqcirc_u^\mu y\} \rangle \in \mathcal{P}$ then $x \eqcirc_u^\mu y \in \mathcal{C}_\infty$.

Proof. We prove that $\langle \mathcal{F}_\infty, \mathcal{C}_\infty \rangle$ is a CSS, meaning that it satisfies properties (P_{css}). Let $(\mathbb{S}\varphi : \mu, x) \in \mathcal{F}_\infty$. We show that $x \simeq x \in \overline{\mathcal{C}_\infty}$ and $\mu \in \mathcal{D}_a(\overline{\mathcal{C}})$. By definition of $\mathcal{F}_\infty$, there is i such that $(\mathbb{S}\varphi : \mu, x) \in \mathcal{F}_i$. By property 3 of Proposition 4.31, $\langle \mathcal{F}_i, \mathcal{C}_i \rangle \in \mathcal{P}$. Then $\langle \mathcal{F}_i, \mathcal{C}_i \rangle$ is a CSS and, by (P_{css}), $x \simeq x \in \overline{\mathcal{C}_i}$ and $\mu \in \mathcal{D}_a(\overline{\mathcal{C}_i})$. Thus, by construction, $x \simeq x \in \overline{\mathcal{C}_\infty}$ and $\mu \in \mathcal{D}_a(\overline{\mathcal{C}_\infty})$. We now prove the properties 1, 2 and 3.

1 Let $\langle \mathcal{F}_f, \mathcal{C}_f \rangle \preccurlyeq_f \langle \mathcal{F}_\infty, \mathcal{C}_\infty \rangle$. As $\mathcal{F}_f$ and $\mathcal{C}_f$ are finite and as the sequence $\langle \mathcal{F}_i, \mathcal{C}_i \rangle_{i \geqslant 0}$ is increasing by property 2 of Proposition 4.31, then there is $j \in \mathbb{N}$ such that $\langle \mathcal{F}_f, \mathcal{C}_f \rangle \preccurlyeq \langle \mathcal{F}_j, \mathcal{C}_j \rangle$. By property 3 of Proposition 4.31, $\langle \mathcal{F}_j, \mathcal{C}_j \rangle \in \mathcal{P}$. As $\mathcal{P}$ is $\preccurlyeq$-closed then we have $\langle \mathcal{F}_f, \mathcal{C}_f \rangle \in \mathcal{P}$. Thus for all $\langle \mathcal{F}_f, \mathcal{C}_f \rangle \preccurlyeq_f \langle \mathcal{F}_\infty, \mathcal{C}_\infty \rangle$, we have $\langle \mathcal{F}_f, \mathcal{C}_f \rangle \in \mathcal{P}$. Therefore $\langle \mathcal{F}_\infty, \mathcal{C}_\infty \rangle \in \mathcal{P}$, because $\mathcal{P}$ is of finite character.
2 Let $(\mathbb{S}\varphi : \mu, x)$ such that $\langle \mathcal{F}_\infty \cup \{(\mathbb{S}\varphi : \mu, x)\}, \mathcal{C}_\infty \rangle \in \mathcal{P}$. By property (P_{css}), $x \simeq x \in \overline{\mathcal{C}_\infty}$ and $\mu \in \mathcal{D}_a(\overline{\mathcal{C}_\infty})$. By Proposition 4.9, $\rhd\mu \in \overline{\mathcal{C}_\infty}$. By compactness (Lemma 4.11), there is $\mathcal{C}_{f_1} \subseteq \mathcal{C}_\infty$ and $\mathcal{C}_{f_2} \subseteq \mathcal{C}_\infty$ such that $\mathcal{C}_{f_1}$ and $\mathcal{C}_{f_2}$ are finite and $x \simeq x \in \overline{\mathcal{C}_{f_1}}$ and $\rhd\mu \in \overline{\mathcal{C}_{f_2}}$. As the sequence is increasing, by property 2 of Proposition 4.31, there is $j \in \mathbb{N}$ such that $(\mathcal{C}_{f_1} \cup \mathcal{C}_{f_2}) \subseteq \mathcal{C}_j$. As $(\mathbb{S}\varphi : \mu, x)$ occurs infinitely many times in our fair strategy $\mathcal{S}$, there is $k \geqslant j$ such that $\mathcal{S}_k = (\mathbb{S}\varphi : \mu, x)$. Moreover $\mathcal{C}_j \subseteq \mathcal{C}_k$. Then $x \simeq x \in \overline{\mathcal{C}_k}$ and $\rhd\mu \in \overline{\mathcal{C}_k}$ (so $\mu \in \mathcal{D}_a(\overline{\mathcal{C}_k})$). Thus $\langle \mathcal{F}_k \cup \{(\mathbb{S}\varphi : \mu, x)\}, \mathcal{C}_k \rangle$ is a CSS (satisfies the property (P_{css})) and $\langle \mathcal{F}_k \cup \{(\mathbb{S}\varphi : \mu, x)\}, \mathcal{C}_k \rangle \preccurlyeq \langle \mathcal{F}_\infty \cup \{(\mathbb{S}\varphi : \mu, x)\}, \mathcal{C}_\infty \rangle$, by definition of limit CSS. As $\mathcal{P}$ is $\preccurlyeq$-closed then $\langle \mathcal{F}_k \cup \{(\mathbb{S}\varphi : \mu, x)\}, \mathcal{C}_k \rangle \in \mathcal{P}$. By construction of $\langle \mathcal{F}_{k+1}, \mathcal{C}_{k+1} \rangle$, $(\mathbb{S}\varphi : \mu, x) \in \mathcal{F}_{k+1}$. Therefore $(\mathbb{S}\varphi : \mu, x) \in \mathcal{F}_\infty$.
3 Let $x \eqcirc_u^\mu y$ such that $\langle \mathcal{F}_\infty, \mathcal{C}_\infty \cup \{x \eqcirc_u^\mu y\} \rangle \in \mathcal{P}$. Let $k = max\{i \in \mathbb{N} \mid c_i \in \mathcal{E}(x) \cup \mathcal{E}(y)\}$. As $x \eqcirc_u^\mu y$ occurs infinitely many times in our fair strategy $\mathcal{S}$, there is $l \geqslant k$ such that $\mathcal{S}_l = x \eqcirc_u^\mu y$. We have $\langle \mathcal{F}_l, \mathcal{C}_l \cup \{x \eqcirc_u^\mu y\} \rangle \preccurlyeq \langle \mathcal{F}_\infty, \mathcal{C}_\infty \cup \{x \eqcirc_u^\mu y\} \rangle$, by definition of limit CSS. As $\mathcal{P}$ is $\preccurlyeq$-closed then $\langle \mathcal{F}_l, \mathcal{C}_l \cup \{x \eqcirc_u^\mu y\} \rangle \in \mathcal{P}$. Moreover, as $\gamma_r \cap (\mathcal{E}(x) \cup \mathcal{E}(y)) \subseteq \{c_1, ..., c_{2l+1}\}$ then, by construction of $\langle \mathcal{F}_{l+1}, \mathcal{C}_{l+1} \rangle$, $x \eqcirc_u^\mu y \in \mathcal{C}_{l+1}$. Therefore $x \eqcirc_u^\mu y \in \mathcal{C}_\infty$.

□

Lemma 4.33. The limit CSS is a Hintikka CSS.

Proof. By property 1 of Proposition 4.32, $\langle \mathcal{F}_\infty, \mathcal{C}_\infty \rangle \in \mathcal{P}$. We verify that all conditions of Definition 4.21 hold. The detailed proof is given in Appendix F. □

Theorem 4.34 (Completeness). Let φ be a formula. If φ is valid then there exits a proof for φ.

Proof. We suppose that there is no proof for the formula φ. We show that φ is not valid. The method that we present here allows us to build a limit CSS $\langle \mathcal{F}_\infty, \mathcal{C}_\infty \rangle$ that is a Hintikka CSS, by Lemma 4.33. By property 1 of Proposition 4.31, $(\mathbb{F}\varphi : [\![]\!], c_1) \in \mathcal{F}_i$ for any $i \geqslant 0$. By definition of limit CSS, $(\mathbb{F}\varphi : [\![]\!], c_1) \in \mathcal{F}_\infty$. By Lemma 4.25, φ is not valid. □

5. Conclusion

In this paper we defined a new logic, called Public Announcement Separation Logic (PASL), extension of the logic ESL [Courtault et al. 2015] with public announcements, with possible worlds considered as resources, and then introduced the sharing and the separation on these worlds. An illustrating example emphasized the power of PASL for modelling and completed this study about expressivity with the proposal of new modalities combining epistemic and separation connectives. To complete this study we also provided a tableau calculus, in the spirit of calculi for BI and BBI [Courtault et al. 2015,Galmiche et al. 2005,Larchey-Wendling 2016], but with specific labels, constraints and rules for resources, agents and mainly for announcements. We proved its soundness and completeness by adapting and improving methods previously used for some modal bunched logics [Courtault and Galmiche 2015,Courtault et al. 2015] and also provided a method for countermodel extraction. An original point is that constraints are decorated with stacks of formulas, knowing that for PAL one only decorates formulas.

Future work will be devoted to the study of semantics and calculi for extensions of this logic that deal with epistemic actions [Baltag et al. 2006]. Extensions with other modalities dealing with dynamic resources [Courtault and Galmiche 2013,Courtault and Galmiche 2015] will also be studied. Our work is also an attempt to enrich some separation logics with uncertainty over composition and decomposition of resources, and by different agents. Here we have studied separation through BBI logic and its resource semantics but we expect to study such enrichments from other resource logics and models with separation, like for instance SL based on memory models and dedicated to program verification [Ishtiaq and O'Hearn 2001,Reynolds 2002].

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Appendix A. Proof of Lemma 4.11

Lemma A.1 (Compactness). Let $\mathcal{C}$ be a (possibly infinite) set of constraints:

1. If $x \simeq y \in \overline{\mathcal{C}}$ then there is a finite set $\mathcal{C}_f$ such that $\mathcal{C}_f \subseteq \mathcal{C}$ and $x \simeq y \in \overline{\mathcal{C}_f}$
2. If $x \mathrel{\dot{=}}^{\mu}_{u} y \in \overline{\mathcal{C}}$ then there is a finite set $\mathcal{C}_f$ such that $\mathcal{C}_f \subseteq \mathcal{C}$ and $x \mathrel{\dot{=}}^{\mu}_{u} y \in \overline{\mathcal{C}_f}$
3. If $\triangleright\mu \in \overline{\mathcal{C}}$ then there is a finite set $\mathcal{C}_f$ such that $\mathcal{C}_f \subseteq \mathcal{C}$ and $\triangleright\mu \in \overline{\mathcal{C}_f}$

Proof. Let $\mathcal{C}$ be a set of constraints. Let $c \in \overline{\mathcal{C}}$ be a constraint. If $c \in \overline{\mathcal{C}}$ because $c \in \mathcal{C}$ then by considering $\mathcal{C}_f = \{c\}$, we have $\mathcal{C}_f \subseteq \mathcal{C}$ and $c \in \overline{\mathcal{C}_f}$. In the other cases, the constraint c is obtained by rules of Fig. 6. We prove the Lemma by induction on the size n of the deduction tree of c.

— Base case ($n = 0$):
  - Case rule $\langle \varepsilon \rangle$: the deduction tree is of the form

$$\frac{}{\varepsilon \simeq \varepsilon}\ \langle \varepsilon \rangle$$

In this case, c is the constraint $\varepsilon \simeq \varepsilon$. Considering $\mathcal{C}_f = \emptyset$, we have $\mathcal{C}_f \subseteq \mathcal{C}$ and $c \in \overline{\mathcal{C}_f}$.

— Inductive step:
We suppose that the properties (1) and (2) hold for deduction trees whose sizes are least or equal to n (IH). We prove the Lemma for deduction trees such that their sizes are equal to $n+1$.
  - Case $\langle r_a \rangle$: the deduction tree is of the form

$$\frac{\dfrac{\vdots}{x \simeq x}}{x \mathrel{\dot{=}}^{[\![]\!]}_{v} x}\ \langle r_a \rangle$$

In this case, c is the constraint $x \mathrel{\dot{=}}^{[\![]\!]}_{v} x$. This deduction tree is finite, and the deduction tree of $x \simeq x$ has a size lower or equal to n. Then, by (IH), there is a finite set $\mathcal{C}_f \subseteq \mathcal{C}$ such that $x \simeq x \in \overline{\mathcal{C}_f}$. Thus, by the rule $\langle r_a \rangle$, $x \mathrel{\dot{=}}^{[\![]\!]}_{v} x \in \overline{\mathcal{C}_f}$.
  - Case $\langle k_a \rangle$: the deduction tree is of the form

$$\frac{\dfrac{\vdots}{x \mathrel{\dot{=}}^{[\![]\!]}_{u} y} \qquad \dfrac{\vdots}{x \simeq k}}{k \mathrel{\dot{=}}^{[\![]\!]}_{u} y}\ \langle k_a \rangle$$

In this case, c is the constraint $k \mathrel{\dot{=}}^{[\![]\!]}_{u} y$. This deduction tree is finite, and the deduction trees of $x \mathrel{\dot{=}}^{[\![]\!]}_{u} y$ and $x \simeq k$ have a size lower or equal to n. Then, by (IH), there are $\mathcal{C}_{f_1} \subseteq \mathcal{C}$ and $\mathcal{C}_{f_2} \subseteq \mathcal{C}$ that are finite and such that $x \mathrel{\dot{=}}^{[\![]\!]}_{u} y \in \overline{\mathcal{C}_{f_1}}$ and $x \simeq k \in \overline{\mathcal{C}_{f_2}}$. Let $\mathcal{C}_f = \mathcal{C}_{f_1} \cup \mathcal{C}_{f_2}$. Then $x \mathrel{\dot{=}}^{[\![]\!]}_{u} y \in \overline{\mathcal{C}_f}$ and $x \simeq k \in \overline{\mathcal{C}_f}$. Thus, by the rule $\langle k_a \rangle$, $k \mathrel{\dot{=}}^{[\![]\!]}_{u} y \in \overline{\mathcal{C}_f}$. Moreover, $\mathcal{C}_f$ is finite as the union of two finite sets and $\mathcal{C}_f \subseteq \mathcal{C}$ as the union of two sets included in $\mathcal{C}$.
  - Case $\langle p_a \rangle$: the deduction tree is of the form

$$\frac{\dfrac{\vdots}{x \mathrel{\dot{=}}^{[\![\psi_1;\ldots;\psi_k]\!]}_{u} y}}{k \mathrel{\dot{=}}^{[\![\psi_1;\ldots;\psi_{k-1}]\!]}_{u} y}\ \langle p_a \rangle$$

In this case, c is the constraint $k =_u^{[\![\psi_1;\ldots;\psi_{k-1}]\!]} y$. This deduction tree is finite, and the deduction tree of $x =_u^{[\![\psi_1;\ldots;\psi_k]\!]} y$ has a size equal to n. Then, by (IH), there is $\mathcal{C}_f \subseteq \mathcal{C}$ that is finite and such that $x =_u^{[\![\psi_1;\ldots;\psi_k]\!]} y \in \overline{\mathcal{C}_f}$. Thus, by the rule $\langle p_a \rangle$, $k =_u^{[\![\psi_1;\ldots;\psi_{k-1}]\!]} y \in \overline{\mathcal{C}_f}$.

- Case $\langle a_n \rangle$:
  Deduction tree is of the form

$$\dfrac{\dfrac{\vdots}{x =_u^{[\![\psi_1;\ldots;\psi_k]\!]} y}}{\triangleright[\![\psi_1;\ldots;\psi_{k-1}]\!]}\ \langle a_n \rangle$$

  In this case, c is the constraint $\triangleright[\![\psi_1;\ldots;\psi_{k-1}]\!]$. This deduction tree is finite, and the deduction tree of $x =_u^{[\![\psi_1;\ldots;\psi_k]\!]}$ has a size equal to n. Then, by (IH), there is $\mathcal{C}_f \subseteq \mathcal{C}$ that is finite and such that $x =_u^{[\![\psi_1;\ldots;\psi_k]\!]} \in \overline{\mathcal{C}_f}$. Thus, by the rule $\langle a_n \rangle$, $\triangleright[\![\psi_1;\ldots;\psi_{k-1}]\!] \in \overline{\mathcal{C}_f}$.

- The other cases are proved similarly.

□

Appendix B. Proof of Proposition 4.17

Proposition B.1. Let $\langle \mathcal{F}, \mathcal{C} \rangle$ be a CSS and $\mathfrak{R} = (\mathcal{M}, |.|)$ a realization of it. $\mathfrak{R}$ is a realization of $\langle \mathcal{F}, \overline{\mathcal{C}} \rangle$, in other words:

1 For all $x \in \mathcal{D}_r(\overline{\mathcal{C}})$, $|x|$ is defined
2 If $x \simeq y \in \overline{\mathcal{C}}$ then $|x| = |y|$
3 If $x =_u^{[\![\psi_1;\ldots;\psi_k]\!]} y \in \overline{\mathcal{C}}$ then $|x| \sim_u |y|$ in the updated model $\mathcal{M}|\psi_1|\ldots|\psi_k$

Proof. We prove the properties for all constraints that belong to $\overline{\mathcal{C}}$. Let $c \in \overline{\mathcal{C}}$ be a constraint. If $c \in \overline{\mathcal{C}}$ because $c \in \mathcal{C}$ then there are two cases:

— c is a constraint of the form $x \simeq y$. In this case, $x \in \mathcal{D}_r(\mathcal{C})$, $y \in \mathcal{D}_r(\mathcal{C})$ and $x \simeq y \in \mathcal{C}$. Then $|x|$ and $|y|$ are defined, by definition of realization, and we have $|x| = |y|$.
— c is a constraint of the form $x =_u^{[\![\psi_1;\ldots;\psi_k]\!]} y$. In this case, $x \in \mathcal{D}_r(\mathcal{C})$, $y \in \mathcal{D}_r(\mathcal{C})$ and $x =_u^{[\![\psi_1;\ldots;\psi_k]\!]} y \in \mathcal{C}$. Then $|x|$ and $|y|$ are defined, and we have $|x| \sim_u |y|$ in the updated model $\mathcal{M}|\psi_1|\ldots|\psi_k$.

Else this constraint is obtained by rules of Fig. 6. We prove by mutual induction on size n of the constraint deduction tree.

— Base case ($n = 0$):
  - Case rule $\langle \varepsilon \rangle$: the deduction tree is of the form

$$\overline{\varepsilon \simeq \varepsilon}\ \langle \varepsilon \rangle$$

  In this case c is the constraint $\varepsilon \simeq \varepsilon$. We can remark that $|\varepsilon|$ is defined ($|\varepsilon| = e$) and $|\varepsilon| = |\varepsilon|$.

— Inductive step:
  We suppose that the Lemma holds for the constraints having a deduction tree whose size is least or equal to n (IH). We prove the Lemma for the constraints having a deduction tree such that their sizes are equal to $n+1$.

– Case $\langle c_r \rangle$: the deduction tree is of the form

$$\dfrac{\dfrac{\vdots}{x \simeq y} \qquad \dfrac{\vdots}{yk \simeq yk}}{xk \simeq yk}\ \langle c_r \rangle$$

By (IH), $|x|$, $|y|$ and $|yk|$ are defined. Again by (IH), $|x| = |y|$. $|yk|$ begin defined, we have, by definition, $|y| \bullet |k| \downarrow$ and $|y| \bullet |k| = |yk|$. Thus $|x| \bullet |k| \downarrow$ and $|x| \bullet |k| = |y| \bullet |k|$. Then $|xk| = |x| \bullet |k| = |y| \bullet |k| = |yk|$. Therefore we have $|xk| = |yk|$.

– Case $\langle k_r \rangle$: the deduction tree is of the form

$$\dfrac{\dfrac{\vdots}{x =_u^{\mu} y}}{x \simeq x}\ \langle k_r \rangle$$

By (IH), $|x|$ is defined and we have obviously $|x| = |x|$.

– Case $\langle r_a \rangle$: the deduction tree is of the form

$$\dfrac{\dfrac{\vdots}{x \simeq x}}{x =_v^{[\![]\!]} x}\ \langle r_a \rangle$$

By (IH), $|x|$ is defined. By reflexivity, $|x| \sim_v |x|$ in the model $\mathcal{M}$.

– Case $\langle k_a \rangle$:
the deduction tree is of the form

$$\dfrac{\dfrac{\vdots}{x =_u^{[\![]\!]} y} \qquad \dfrac{\vdots}{x \simeq k}}{k =_u^{[\![]\!]} y}\ \langle k_a \rangle$$

By (IH), $|x|$, $|y|$ and $|k|$ are defined and we have $|x| \sim_u |y|$ in the model $\mathcal{M}$ and $|x| = |k|$. Therefore, we have $|k| \sim_u |x|$ in the model $\mathcal{M}$.

– Case $\langle p_a \rangle$: the deduction tree is of the form

$$\dfrac{\dfrac{\vdots}{x =_u^{[\![\psi_1;\ldots;\psi_k]\!]} y}}{x =_u^{[\![\psi_1;\ldots;\psi_{k-1}]\!]} x}\ \langle p_a \rangle$$

By (IH), $|x|$ and $|y|$ are defined. Moreover, again by (IH), $|x| \sim_u |y|$ in the updated model $\mathcal{M}|\psi_1|\ldots|\psi_k$. Then, by definition, $|x| \sim_u |y|$ in the updated model $\mathcal{M}|\psi_1|\ldots|\psi_{k-1}$.

– The other cases are proved similarly.

□

Appendix C. Proof of Lemma 4.18

Lemma C.1. Rules of the PASL tableaux calculus preserve realizability.

Proof. Let $\mathcal{T}$ be a realizable tableau. By definition, $\mathcal{T}$ contains a realizable branch $\mathcal{B} = \langle \mathcal{F}, \mathcal{C} \rangle$. Let $\mathfrak{R} = (\mathcal{M}, |.|)$ a realization of the branch $\mathcal{B}$, where $\mathcal{M} = (\mathcal{R}, \{\sim_a\}_{a \in A}, V)$ and $|.| : \mathcal{D}_r(\mathcal{C}) \to R$. If we apply a rule on a labelled formula of another branch than $\mathcal{B}$ then $\mathcal{B}$ is not modified and $\mathcal{T}$ stays realizable. Else, we prove by case on the formula or agent constraint whose is applied the rule.

— $(\mathbb{T}\mathrm{I} : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$:
We have, by definition of realization, $|x| \vDash_{\mathcal{M}|\psi_1|...|\psi_k} \mathrm{I}$. Then $|x| = e$. As $|\varepsilon| = e$ then $|x| = |\varepsilon|$ and we remark that $\mathfrak{R}$ is a realization of the new branch $\langle \mathcal{F}, \mathcal{C} \cup \{x \simeq \varepsilon\} \rangle$.

— $(\mathbb{T}\varphi_1 * \varphi_2 : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$:
By realization, we have $|x| \vDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_1 * \varphi_2$. Then, by definition, there exists $r_1, r_2 \in R$ such that $r_1 \bullet r_2 \downarrow$, $|x| = r_1 \bullet r_2$, $r_1 \vDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_1$ and $r_2 \vDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_2$. As c_i and c_j are new resource label constants, $|c_i|$ and $|c_j|$ are not defined. Moreover as $c_i \neq c_j$, we can extend $\mathfrak{R}$ by setting $|c_i| = r_1$ and $|c_j| = r_2$. Remarking that $|c_i| \bullet |c_j| \downarrow$ and, by implicit extension, $|x| = |c_i| \bullet |c_j| = |c_i c_j|$, we obtain a realization of $\langle \mathcal{F}, \mathcal{C} \cup \{x \simeq c_i c_j\} \rangle$. Moreover, this realization is a realization of the new branch $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi_1 : [\![\psi_1;...;\psi_k]\!], c_i), (\mathbb{T}\varphi_2 : [\![\psi_1;...;\psi_k]\!], c_j)\}, \mathcal{C} \cup \{x \simeq c_i c_j\} \rangle$.

— $(\mathbb{F}\varphi_1 * \varphi_2 : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$:
We have $|x| \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_1 * \varphi_2$. By definition, for all $r_1, r_2 \in R$ such that $r_1 \bullet r_2 \downarrow$ and $|x| = r_1 \bullet r_2$, we have $r_1 \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi$ or $r_2 \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} \psi$. The branch is expanded into two branches that are $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : [\![\psi_1;...;\psi_k]\!], y)\}, \mathcal{C} \rangle$ and $\langle \mathcal{F} \cup \{(\mathbb{F}\psi : [\![\psi_1;...;\psi_k]\!], z)\}, \mathcal{C} \rangle$ where $x \simeq yz \in \overline{\mathcal{C}}$. By Proposition 4.17, $|x| = |yz|$. By definition of realization, $|.|$ is total, so $|y| \bullet |z| \downarrow$ and $|yz| = |y| \bullet |z|$. Thus $|y| \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi$ or $|z| \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} \psi$. Therefore $\mathfrak{R}$ is a realization of at least one of the two new branches $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : [\![\psi_1;...;\psi_k]\!], y)\}, \mathcal{C} \rangle$ or $\langle \mathcal{F} \cup \{(\mathbb{F}\psi : [\![\psi_1;...;\psi_k]\!], z)\}, \mathcal{C} \rangle$.

— $(\mathbb{T}K_a\varphi : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$:
We have $|x| \vDash_{\mathcal{M}|\psi_1|...|\psi_k} K_a\varphi$. Then, by definition, for all $r \in R$ such that $|x| \sim_a r$ in the updated model $\mathcal{M}|\psi_1|...|\psi_k$, we have $r \vDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi$. By rule condition, $x \eqsim_a^{[\![\psi_1;...;\psi_k]\!]} y \in \overline{\mathcal{C}}$. Thus, by Proposition 4.17, we have $|x| \sim_a |y|$ in the updated model $\mathcal{M}|\psi_1|...|\psi_k$. Therefore $|y| \vDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi$ and we can conclude that $\mathfrak{R}$ is a realization of the new branch $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : [\![\psi_1;...;\psi_k]\!], y)\}, \mathcal{C} \rangle$.

— $(\mathbb{F}K_a\varphi : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$:
By realization, we have $|x| \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} K_a\varphi$. Then, there is $r \in R$ such that $|x| \sim_a r$ in the updated model $\mathcal{M}|\psi_1|...|\psi_k$ and $r \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi$. As c_i is a new label constants, then $|c_i|$ is not defined. Then, we can extend $\mathfrak{R}$ such that $|c_i| = r$. Remarking that $|x| \sim_a |c_i|$ in the model $\mathcal{M}|\psi_1|...|\psi_k$, we obtain a realization of $\langle \mathcal{F}, \mathcal{C} \cup \{x \eqsim_a^{[\![\psi_1;...;\psi_k]\!]} c_i\} \rangle$. Moreover, this extension is obviously a realization of the new branch $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : [\![\psi_1;...;\psi_k]\!], c_i)\}, \mathcal{C} \cup \{x \eqsim_a^{[\![\psi_1;...;\psi_k]\!]} c_i\} \rangle$.

— $(\mathbb{T}[\varphi_1]\varphi_2 : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$:
By realization $|x| \vDash_{\mathcal{M}|\psi_1|...|\psi_k} [\varphi_1]\varphi_2$. Then, if $|x| \vDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_1$ then $|x| \vDash_{\mathcal{M}|\psi_1|...|\psi_k|\varphi_1} \varphi_2$. There are two cases:

- Case $|x| \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_1$:

In this case, we observe that $\mathfrak{R}$ is a realization of the first new branch $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi_1 : [\![\psi_1;...;\psi_k]\!], x)\}, \mathcal{C}\rangle$.

- Case $|x| \vDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_1$:
  Then we have $|x| \vDash_{\mathcal{M}|\psi_1|...|\psi_k|\varphi_1} \varphi_2$ and we remark that $\mathfrak{R}$ is a realization of the second new branch $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi_2 : [\![\psi_1;...;\psi_k;\varphi_1]\!], x)\}, \mathcal{C} \cup \{\triangleright[\![\psi_1;...;\psi_k;\varphi_1]\!]\}\rangle$.

— $(\mathbb{F}[\varphi_1]\varphi_2 : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$:
By realization $|x| \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} [\varphi_1]\varphi_2$. Then $|x| \vDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_1$ and $|x| \nvDash_{\mathcal{M}|\psi_1|...|\psi_k|\varphi_1} \varphi_2$. Thus we remark that $\mathfrak{R}$ is a realization of the new branch $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi_1 : [\![\psi_1;...;\psi_k]\!], x), (\mathbb{F}\varphi_2 : [\![\psi_1;...;\psi_k;\varphi_1]\!], x)\}, \mathcal{C} \cup \{\triangleright[\![\psi_1;...;\psi_k;\varphi_1]\!]\}\rangle$.

— $x \asymp_u^{[\![\psi_1;...;\psi_k]\!]} y \in \overline{\mathcal{C}}$:
By Proposition 4.17, $|x| \sim_u |y|$ in the updated model $\mathcal{M}|\psi_1|...|\psi_k$. By definition, we have $|x| \vDash_{\mathcal{M}|\psi_1|...|\psi_{k-1}} \psi_k$ iff $|y| \vDash_{\mathcal{M}|\psi_1|...|\psi_{k-1}} \psi_k$. There are two cases:

- Case $|x| \vDash_{\mathcal{M}|\psi_1|...|\psi_{k-1}} \psi_k$:
  In this case, we have $|y| \vDash_{\mathcal{M}|\psi_1|...|\psi_{k-1}} \psi_k$. Thus $\mathfrak{R}$ is a realization of the first new branch $\langle \mathcal{F} \cup \{(\mathbb{T}\psi_k : [\![\psi_1;...;\psi_{k-1}]\!], x), (\mathbb{T}\psi_k : [\![\psi_1;...;\psi_{k-1}]\!], y)\}, \mathcal{C}\rangle$.
- Case $|x| \nvDash_{\mathcal{M}|\psi_1|...|\psi_{k-1}} \psi_k$:
  In this case, we have $|y| \nvDash_{\mathcal{M}|\psi_1|...|\psi_{k-1}} \psi_k$. Thus $\mathfrak{R}$ is a realization of the second new branch $\langle \mathcal{F} \cup \{(\mathbb{F}\psi_k : [\![\psi_1;...;\psi_{k-1}]\!], x), (\mathbb{F}\psi_k : [\![\psi_1;...;\psi_{k-1}]\!], y)\}, \mathcal{C}\rangle$.

— $x \asymp_u^{[\![\psi_1;...;\psi_k]\!]} y \in \overline{\mathcal{C}}$ and $\triangleright[\![\psi_1;...;\psi_k;\varphi]\!] \in \overline{\mathcal{C}}$:
By Proposition 4.17, $|x| \sim_u |y|$ in the updated model $\mathcal{M}|\psi_1|...|\psi_k$. There are two cases:

- $|x| \sim_u |y|$ in the updated model $\mathcal{M}|\psi_1|...|\psi_k|\varphi$:
  In this case $\mathfrak{R}$ is a realization of the first new branch $\langle \mathcal{F}, \mathcal{C} \cup \{x \asymp_u^{[\![\psi_1;...;\psi_k;\varphi]\!]} y\}\rangle$.
- $|x| \sim_u |y|$ does not hold in the updated model $\mathcal{M}|\psi_1|...|\psi_k|\varphi$:
  There are, by definition, three cases:
  - $|x| \sim_u |y|$ does not hold in the updated model $\mathcal{M}|\psi_1|...|\psi_k$:
    This case is absurd.
  - $|x| \vDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi$ and $|y| \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi$:
    Thus $\mathfrak{R}$ is a realization of the second new branch $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : [\![\psi_1;...;\psi_k]\!], x), (\mathbb{F}\varphi : [\![\psi_1;...;\psi_k]\!], y)\}, \mathcal{C}\rangle$.
  - $|x| \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi$ and $|y| \vDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi$:
    Thus $\mathfrak{R}$ is a realization of the second new branch $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : [\![\psi_1;...;\psi_k]\!], x), (\mathbb{T}\varphi : [\![\psi_1;...;\psi_k]\!], y)\}, \mathcal{C}\rangle$.

— Other cases are proved similarly.

□

Appendix D. Proof of Lemma 4.24

Lemma D.1. Let $\langle \mathcal{F}, \mathcal{C}\rangle$ be a Hintikka CSS and $\mathcal{M} = \Omega(\langle \mathcal{F}, \mathcal{C}\rangle) = (\mathcal{R}, \{\sim_a\}_{a \in A}, V)$, where $\mathcal{R} = (R, \bullet, e)$. For all formulae $\varphi \in \mathcal{L}$, all $[\![\psi_1;...;\psi_k]\!] \in \mathcal{D}_a(\overline{\mathcal{C}})$, all agents $a \in A$ and all $x, y \in \mathcal{D}_r(\overline{\mathcal{C}})$, we have

(1) $x \asymp_a^{[\![\psi_1;...;\psi_k]\!]} y \in \overline{\mathcal{C}}$ iff $[x] \sim_a [y]$ in the model $\mathcal{M}|\psi_1|...|\psi_k$

(2) If $(\mathbb{F}\varphi : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$ then $[x] \not\models_{\mathcal{M}|\psi_1|...|\psi_k} \varphi$
(3) If $(\mathbb{T}\varphi : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$ then $[x] \models_{\mathcal{M}|\psi_1|...|\psi_k} \varphi$

Proof. These properties are proved simultaneously by induction on $len(x \doteq_a^{[\![\psi_1;...;\psi_k]\!]} y)$ and $len(\mathcal{M}|\psi_1|...|\psi_k)$ for the property (1) and on $len(\mathbb{S}\varphi : \mu, x)$ for the properties (2) and (3), where for len see Definition 4.24 on page 22:

Base case ($len(X) = 2$)

— Property (1):
  – Let $x \doteq_a^{[\![]\!]} y \in \overline{\mathcal{C}}$. By definition of Ω, $[x] \sim_a [y]$ in the model $\mathcal{M}$.
  – We suppose now$[x] \sim_a [y]$ in the model $\mathcal{M}$. Then, again by definition of Ω, we have $x \doteq_a^{[\![]\!]} y \in \overline{\mathcal{C}}$.

— Case $(\mathbb{F}p : [\![]\!], x) \in \mathcal{F}$ such that $p \in Prop$:
We suppose that $[x] \models_{\mathcal{M}} p$. Then $[x] \in V(p)$. By definition Ω, there are a resource label y and formulae $\psi'_1, ..., \psi'_l \in \mathcal{L}$ such that we have $y \simeq x \in \overline{\mathcal{C}}$ and $(\mathbb{T}p : [\![\psi'_1;...;\psi'_l]\!], y) \in \mathcal{F}$. By condition (1) of Definition 4.21, $\langle \mathcal{F}, \mathcal{C} \rangle$ is not a Hintikka CSS. This is absurd, and then $[x] \not\models_{\mathcal{M}} p$.

— Case $(\mathbb{T}p : [\![]\!], x) \in \mathcal{F}$ such that $p \in Prop$:
By property by (P_{css}), $x \simeq x \in \overline{\mathcal{C}}$. Then, by definition of Ω, $[x] \in V(p)$. Thus $[x] \models_{\mathcal{M}} p$.

— Case $(\mathbb{F}\bot : [\![]\!], x) \in \mathcal{F}$:
We have $[x] \not\models_{\mathcal{M}} \bot$, by definition.

— Case $(\mathbb{T}\bot : [\![]\!], x) \in \mathcal{F}$:
As $\langle \mathcal{F}, \mathcal{C} \rangle$ is a Hintikka CSS then, by condition (5) of Definition 4.21, this case is absurd.

— Case $(\mathbb{F}\mathrm{I} : [\![]\!], x) \in \mathcal{F}$:
We suppose that $[x] \models_{\mathcal{M}} \mathrm{I}$. Then $[x] = e$ and, by definition of Ω, we have $[x] = [\varepsilon]$. Therefore $x \simeq \varepsilon \in \overline{\mathcal{C}}$. Then, by condition (3) of Definition 4.21, $\langle \mathcal{F}, \mathcal{C} \rangle$ is not a Hintikka CSS. It is absurd and we can conclude that $[x] \not\models_{\mathcal{M}} \mathrm{I}$.

— Case $(\mathbb{T}\mathrm{I} : [\![]\!], x) \in \mathcal{F}$:
By condition (3) of Definition 4.21, $x \simeq \varepsilon \in \overline{\mathcal{C}}$. Then, by definition of Ω, $[x] = [\varepsilon] = e$. Therefore $[x] \models_{\mathcal{M}} \mathrm{I}$.

— The other base cases ($(\mathbb{F}\top : [\![]\!], x) \in \mathcal{F}$ and $(\mathbb{T}\top : [\![]\!], x) \in \mathcal{F}$) are treated similarly.

Inductive step: we suppose that the properties (1), (2) and (3) hold for $len(X) \leq i$ (IH) and show the properties for $len(X) = i + 1$.

— Property (1):
  – Let $x \doteq_a^{[\![\psi_1;...;\psi_k]\!]} y \in \overline{\mathcal{C}}$. By condition (27) of Definition 4.21, $(\mathbb{T}\psi_k : [\![\psi_1;...;\psi_{k-1}]\!], x) \in \mathcal{F}$ and $(\mathbb{T}\psi_k : [\![\psi_1;...;\psi_{k-1}]\!], y) \in \mathcal{F}$, or we have $(\mathbb{F}\psi_k : [\![\psi_1;...;\psi_{k-1}]\!], x) \in \mathcal{F}$ and $(\mathbb{F}\psi_k : [\![\psi_1;...;\psi_{k-1}]\!], y) \in \mathcal{F}$. By (IH), we have $[x] \models_{\mathcal{M}|\psi_1|...|\psi_{k-1}} \psi_k$ and $[y] \models_{\mathcal{M}|\psi_1|...|\psi_{k-1}} \psi_k$, or $[x] \not\models_{\mathcal{M}|\psi_1|...|\psi_{k-1}} \psi_k$ and $[y] \not\models_{\mathcal{M}|\psi_1|...|\psi_{k-1}} \psi_k$. Moreover, by rule $\langle p_a \rangle$, $x \doteq_a^{[\![\psi_1;...;\psi_{k-1}]\!]} y \in \overline{\mathcal{C}}$, and by (IH), we have $[x] \sim_a [y]$ in the model $\mathcal{M}|\psi_1|...|\psi_{k-1}$. Thus, we have $[x] \sim_a [y]$ in the model $\mathcal{M}|\psi_1|\psi_k$.
  – We suppose now that $[x] \sim_a [y]$ in the model $\mathcal{M}|\psi_1|...|\psi_k$ (and that $[\![\psi_1;...;\psi_k]\!] \in \mathcal{D}_a(\overline{\mathcal{C}})$).

By definition, we have $[x] \sim_a [y]$ in the model $\mathcal{M}|\psi_1|...|\psi_{k-1}$, $[x] \vDash_{\mathcal{M}|\psi_1|...|\psi_{k-1}} \psi_k$ and $[y] \vDash_{\mathcal{M}|\psi_1|...|\psi_{k-1}} \psi_k$, or $[x] \nvDash_{\mathcal{M}|\psi_1||\psi_{k-1}} \psi_k$ and $[y] \nvDash_{\mathcal{M}|\psi_1|...|\psi_{k-1}} \psi_k$. By (IH) and by Proposition 4.9, we have $x \doteq_a^{[\![\psi_1;...;\psi_{k-1}]\!]} y \in \overline{\mathcal{C}}$ and $\triangleright[\![\psi_1;...;\psi_k]\!] \in \overline{\mathcal{C}}$. Then by condition (28) of Definition 4.21, there are three cases:
- $x \doteq_u^{[\![\psi_1;...;\psi_k]\!]} y \in \overline{\mathcal{C}}$.
- $(\mathbb{T}\psi_k : [\![\psi_1;...;\psi_{k-1}]\!], x) \in \mathcal{F}$ and $(\mathbb{F}\psi_k : [\![\psi_1;...;\psi_{k-1}]\!], y) \in \mathcal{F}$: this case is absurd, because by (IH), we would have $[x] \vDash_{\mathcal{M}|\psi_1|..|\psi_{k-1}} \psi_k$ and $[y] \nvDash_{\mathcal{M}|\psi_1|...|\psi_{k-1}} \psi_k$.
- $(\mathbb{F}\psi_k : [\![\psi_1;...;\psi_{k-1}]\!], x) \in \mathcal{F}$ and $(\mathbb{T}\psi_k : [\![\psi_1;...;\psi_{k-1}]\!], y) \in \mathcal{F}$: this case is absurd, because by (IH), we would have $[x] \nvDash_{\mathcal{M}|\psi_1|...|\psi_{k-1}} \psi_k$ and $[y] \vDash_{\mathcal{M}|\psi_1|...|\psi_{k-1}} \psi_k$.
In conclusion, we have $x \doteq_a^{[\![\psi_1;...;\psi_k]\!]} y \in \overline{\mathcal{C}}$ iff $[x] \sim_a [y]$ in the model $\mathcal{M}|\psi_1|...|\psi_k$.

— Case $(\mathbb{F}\varphi_1 \wedge \varphi_2 : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$:
By condition (10) of Definition 4.21, $(\mathbb{F}\varphi_1 : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$ or $(\mathbb{F}\varphi_2 : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$. Then, by (IH), $[x] \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_1$ or $[x] \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_2$. Thus $[x] \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_1 \wedge \varphi_2$.

— Case $(\mathbb{T}\varphi_1 \wedge \varphi_2 : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$:
By condition (9) of Definition 4.21, $(\mathbb{T}\varphi_1 : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$ and $(\mathbb{T}\varphi_2 : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$. Then, by (IH), $[x] \vDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_1$ and $[x] \vDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_2$. Thus $[x] \vDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_1 \wedge \varphi_2$.

— Case $(\mathbb{F}\varphi_1 * \varphi_2 : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$:
Let $r_1, r_2 \in R$ such that $r_1 \bullet r_2 \downarrow$ and $[x] = r_1 \bullet r_2$. By definition of Ω, there is $yz \in \mathcal{D}_r(\overline{\mathcal{C}})$ such that $r_1 = [y]$, $r_2 = [z]$ and $[x] = [y] \bullet [z] = [yz]$. Then $x \simeq yz \in \overline{\mathcal{C}}$ and by condition (16) of Definition 4.21, $(\mathbb{F}\varphi_1 : [\![\psi_1;...;\psi_k]\!], y) \in \mathcal{F}$ or $(\mathbb{F}\varphi_2 : [\![\psi_1;...;\psi_k]\!], z) \in \mathcal{F}$. Then, by (IH), $r_1 \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_1$ or $r_2 \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_2$. Therefore $[x] \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_1 * \varphi_2$.

— Case $(\mathbb{T}\varphi_1 * \varphi_2 : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$:
By condition (15) of Definition 4.21, there are two resource labels y and z such that $x \simeq yz \in \overline{\mathcal{C}}$, $(\mathbb{T}\varphi_1 : [\![\psi_1;...;\psi_k]\!], y) \in \mathcal{F}$ and $(\mathbb{T}\varphi_2 : [\![\psi_1;...;\psi_k]\!], z) \in \mathcal{F}$. By definition of Ω and by (IH), we have $[y] \bullet [z] \downarrow$, $[x] = [yz] = [y] \bullet [z]$ and $[y] \vDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_1$ and $[z] \vDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_2$. Therefore $[x] \vDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_1 * \varphi_2$.

— Case $(\mathbb{F}K_u\varphi : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$:
By condition (20) of Definition 4.21, there is a resource label y such that we have $x \doteq_u^{[\![\psi_1;...;\psi_k]\!]} y \in \overline{\mathcal{C}}$ and $(\mathbb{F}\varphi : [\![\psi_1;...;\psi_k]\!], y) \in \mathcal{F}$. Then, by (IH), there is a resource $[y]$ such that $[x] \sim_u [y]$ in the model $\mathcal{M}|\psi_1|...|\psi_k$ and $[y] \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi$. Therefore, we have $[x] \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} K_u\varphi$.

— Case $(\mathbb{T}K_u\varphi : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$:
Let $r \in R$ such that $[x] \sim_u r$ in the model $\mathcal{M}|\psi_1|...|\psi_k$. By (IH), there is resource label y such that $y \in \mathcal{D}_r(\overline{\mathcal{C}})$, $r = [y]$ and $x \doteq_u^{[\![\psi_1;...;\psi_k]\!]} y \in \overline{\mathcal{C}}$. Thus, by condition (19) of Definition 4.21, $(\mathbb{T}\varphi : [\![\psi_1;...;\psi_k]\!], y) \in \mathcal{F}$. Then, by (IH), $r \vDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi$. Therefore $[x] \vDash_{\mathcal{M}|\psi_1|...|\psi_k} K_u\varphi$.

— Case $(\mathbb{F}[\varphi_1]\varphi_2 : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$:
By condition (24) of Definition 4.21, $(\mathbb{T}\varphi_1 : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$, $(\mathbb{F}\varphi_2 : [\![\psi_1;...;\psi_k;\varphi_1]\!], x) \in \mathcal{F}$. By (IH), $[x] \vDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_1$ and $[x] \nvDash_{\mathcal{M}|\psi_1|...|\psi_k|\varphi_1} \varphi_2$. Therefore $[x] \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} [\varphi_1]\varphi_2$.

— Case $(\mathbb{T}[\varphi_1]\varphi_2 : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$:
We suppose that $[x] \vDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_1$ and we show that $[x] \vDash_{\mathcal{M}|\psi_1|...|\psi_k|\varphi_1} \varphi_2$. By condition (23) of Definition 4.21, there are two cases:
- $(\mathbb{F}\varphi_1 : [\![\psi_1;...;\psi_k]\!], x) \in \mathcal{F}$: By (IH), we have $[x] \nvDash_{\mathcal{M}|\psi_1|...|\psi_k} \varphi_1$, which is absurd.
- $(\mathbb{T}\varphi_2 : [\![\psi_1;...;\psi_k;\varphi_1]\!], x) \in \mathcal{F}$: By (IH), we have $[x] \vDash_{\mathcal{M}|\psi_1|...|\psi_k|\varphi_1} \varphi_2$.
In conclusion, $[x] \vDash_{\mathcal{M}|\psi_1|...|\psi_k} [\varphi_1]\varphi_2$.

— Other cases are proved similarly.

□

Appendix E. Proof of Proposition 4.31

Proposition E.1. For any $i \in \mathbb{N}$, the following properties hold:

1 $(\mathbb{F}\varphi : [\![]\!], c_1) \in \mathcal{F}_i$ and $c_1 \simeq c_1 \in \mathcal{C}_i$
2 $\mathcal{F}_i \subseteq \mathcal{F}_{i+1}$ and $\mathcal{C}_i \subseteq \mathcal{C}_{i+1}$
3 $\langle \mathcal{F}_i, \mathcal{C}_i \rangle_{i \geqslant 0} \in \mathcal{P}$
4 $\mathcal{A}_r(\mathcal{C}_i) \subseteq \{c_1, c_2, \ldots, c_{2i+1}\}$

Proof.

1 This property holds for $i = 0$. As $\langle \mathcal{F}_{i+1}, \mathcal{C}_{i+1} \rangle$ is an extension ($\cup$) of $\langle \mathcal{F}_i, \mathcal{C}_i \rangle$ then this property also holds for all $i \geqslant 0$.
2 This property holds because $\langle \mathcal{F}_{i+1}, \mathcal{C}_{i+1} \rangle$ is an extension ($\cup$) of $\langle \mathcal{F}_i, \mathcal{C}_i \rangle$.
3, 4. We prove the properties 3 and 4 simultaneously by induction on i.

- The base case ($i = 0$) obviously holds: as $\langle \mathcal{F}_0, \mathcal{C}_0 \rangle = \langle \{(\mathbb{F}\varphi : \mu, c_1)\}, \{c_1 \simeq c_1\} \rangle$ then we remark that the property 4 holds and property 3 holds by hypothesis.
- We prove now the inductive case. We suppose that the properties 3 and 4 hold for $i = n$ (IH) and show that they hold for $i = n+1$, by case on the form of S_i.

If S_n is a labelled formula of the form $(\mathbb{S}F : \mu, x)$ then

— If $\langle \mathcal{F}_n \cup \{(\mathbb{S}F : \mu, x)\}, \mathcal{C}_n \rangle \notin \mathcal{P}$ then $\langle \mathcal{F}_{n+1}, \mathcal{C}_{n+1} \rangle = \langle \mathcal{F}_n, \mathcal{C}_n \rangle$. Then the properties 3 and 4 hold by (IH).

— If $\langle \mathcal{F}_n \cup \{(\mathbb{S}F : \mu, x)\}, \mathcal{C}_n \rangle \in \mathcal{P}$ then it is a CSS (the elements of $\mathcal{P}$ are CSS, by definition). Then, by (P_{css}), $x_n \simeq x_n \in \overline{\mathcal{C}_n}$. Thus, we have $\gamma_r \cap \mathcal{E}(x_n) \subseteq \mathcal{A}_r(\overline{\mathcal{C}_n})$. Therefore, by Proposition 4.10, $\gamma_r \cap \mathcal{E}(x_n) \subseteq \mathcal{A}_r(\mathcal{C}_n)$ **(1)**. There are six cases:

- If $\mathbb{S}_n = \mathbb{T}$ and $F_n = \mathrm{I}$:
In this case, $\langle \mathcal{F}_{n+1}, \mathcal{C}_{n+1} \rangle = \langle \mathcal{F}_n \cup \{(\mathbb{S}F : \mu, x)\}, \mathcal{C}_n \cup \{x_n \simeq \varepsilon\} \rangle$. By saturation of $\mathcal{P}$, applying the rule $\langle \mathbb{T}I \rangle$, we have $\langle \mathcal{F}_{n+1}, \mathcal{C}_{n+1} \rangle \in \mathcal{P}$. Then property 3 holds. By **(1)**, we remark that $\mathcal{A}_r(\mathcal{C}_{n+1}) = \mathcal{A}_r(\mathcal{C}_n)$. Then, by (IH), the property 4 holds.

- Case $\mathbb{S}_n = \mathbb{T}$ and $F_n = \varphi * \psi$:
$\langle \mathcal{F}_{n+1}, \mathcal{C}_{n+1} \rangle = \langle \mathcal{F}_n \cup \{(\mathbb{S}F : \mu, x)\} \cup \{(\mathbb{T}\varphi : \mu, c_{2n+2}), (\mathbb{T}\psi : \mu, c_{2n+3})\}, \mathcal{C}_n \cup \{x \simeq c_{2n+2}c_{2n+3}\} \rangle$.
By (IH), $c_{2n+2} \notin \mathcal{A}_r(\mathcal{C}_n)$ and $c_{2n+3} \notin \mathcal{A}_r(\mathcal{C}_n)$, then they are new resource label constants. Moreover, as $\langle \mathcal{F}_n \cup \{(\mathbb{S}F : \mu, x)\}, \mathcal{C}_n \rangle \in \mathcal{P}$ then, by saturation for rule $\langle \mathbb{T}* \rangle$ and using the labels c_{2n+2} and c_{2n+3}, $\langle \mathcal{F}_{n+1}, \mathcal{C}_{n+1} \rangle \in \mathcal{P}$. Thus property 3 holds. Moreover, by **(1)**, $\mathcal{A}_r(\mathcal{C}_{n+1}) = \mathcal{A}_r(\mathcal{C}_n) \cup \{c_{2n+2}, c_{2n+3}\}$. Then, by (IH), the property 4 holds.

- Case $\mathbb{S}_n = \mathbb{F}$ and $F_n = \varphi \mathrel{-\!\!*} \psi$:
This case is proved similarly.

- Case $\mathbb{S}_n = \mathbb{F}$ and $F_n = K_u\varphi$:
In this case $\langle \mathcal{F}_{n+1}, \mathcal{C}_{n+1} \rangle = \langle \mathcal{F}_n \cup \{(\mathbb{S}F : \mu, x)\} \cup \{(\mathbb{F}\varphi : c_{2n+2}, \}), \mathcal{C}_n \cup \{x_n \sim^{\mu}_{u} c_{2n+2}\} \rangle$. By (IH), $c_{2n+2} \notin \mathcal{A}_r(\mathcal{C}_n)$, then it is a resource label constant. As $\langle \mathcal{F}_n \cup \{(\mathbb{S}F : \mu, x)\}, \mathcal{C}_n \rangle \in \mathcal{P}$ then, by saturation for rule $\langle \mathbb{F}K_u \rangle$ and using the label c_{2n+2}, $\langle \mathcal{F}_{n+1}, \mathcal{C}_{n+1} \rangle \in \mathcal{P}$. Thus property 3 holds. Moreover, by **(1)**, $\mathcal{A}_r(\mathcal{C}_{n+1}) = \mathcal{A}_r(\mathcal{C}_n) \cup \{c_{2n+2}\}$. Then, by (IH), the property

4 holds.
- Case $\mathbb{S}_n = \mathbb{T}$ and $F_n = \widetilde{K}_u\varphi$:
This case is proved similarly.
- In the last case, $\langle \mathcal{F}_{n+1}, \mathcal{C}_{n+1}\rangle = \langle \mathcal{F}_n \cup \{(\mathbb{S}F : \mu,x)\}, \mathcal{C}_n\rangle$. By hypothesis, $\langle \mathcal{F}_n \cup \{(\mathbb{S}F : \mu,x)\}, \mathcal{C}_i\rangle \in \mathcal{P}$, then property 3 holds. The property 4 holds by (IH), because $\mathcal{A}_r(\mathcal{C}_{n+1}) = \mathcal{A}_r(\mathcal{C}_n)$.

If S_n is an agent constraint of the form $x \simeq^{\mu}_{u} y$ then

— If $\gamma_r \cap (\mathcal{E}(x) \cup \mathcal{E}(y)) \not\subseteq \{c_1, ..., c_{2n+1}\}$ then $\langle \mathcal{F}_{n+1}, \mathcal{C}_{n+1}\rangle = \langle \mathcal{F}_n, \mathcal{C}_n\rangle$. Then the properties 3 and 4 hold by (IH).

— If $\langle \mathcal{F}_i, \mathcal{C}_i \cup \{x \simeq^{\mu}_{u} y\}\rangle \notin \mathcal{P}$ then $\langle \mathcal{F}_{n+1}, \mathcal{C}_{n+1}\rangle = \langle \mathcal{F}_n, \mathcal{C}_n\rangle$. Then the properties 3 and 4 hold by (IH).

— If $\langle \mathcal{F}_n, \mathcal{C}_n \cup \{x \simeq^{\mu}_{u} y\}\rangle \in \mathcal{P}$ then $\langle \mathcal{F}_{n+1}, \mathcal{C}_{n+1}\rangle = \langle \mathcal{F}_n, \mathcal{C}_n \cup \{x \simeq^{\mu}_{u} y\}\rangle$, so the property 3 holds. As $\gamma_r \cap (\mathcal{E}(x) \cup \mathcal{E}(y)) \subseteq \{c_1, ..., c_{2n+1}\}$, then the property 4 holds by (IH).

□

Appendix F. Proof of Lemma 4.33

Lemma F.1. The limit CSS is a Hintikka CSS.

Proof. By property 1 of Proposition 4.32, $\langle \mathcal{F}_\infty, \mathcal{C}_\infty\rangle \in \mathcal{P}$. We verify that all conditions of Definition 4.21 hold.

1 We suppose that $(\mathbb{T}p : \mu,x) \in \mathcal{F}_\infty$ and $(\mathbb{F}p : \kappa,y) \in \mathcal{F}_\infty$ and $x \simeq y \in \overline{\mathcal{C}_\infty}$. Then $\langle \mathcal{F}_\infty, \mathcal{C}_\infty\rangle$ is closed. Thus, by definition of oracle, $\langle \mathcal{F}_\infty, \mathcal{C}_\infty\rangle \notin \mathcal{P}$. This is absurd, by Proposition 4.32. Then condition 1 of Definition 4.21 holds.
2 This case is similar to condition 1.
3 This case is similar to condition 1.
4 This case is similar to condition 1.
5 This case is similar to condition 1.
6 We suppose that $(\mathbb{T}\mathrm{I} : \mu,x) \in \mathcal{F}_\infty$. Then there is $j \in \mathbb{N}$ such that $(\mathbb{T}\mathrm{I} : \mu,x) \in \mathcal{F}_j$. Moreover there exists $k \geqslant j$ such that $\mathcal{S}_k = (\mathbb{T}\mathrm{I} : \mu,x)$. As the sequence $\langle \mathcal{F}_i, \mathcal{C}_i\rangle_{i \geqslant 0}$ is increasing (property 2 of Proposition 4.31), then $(\mathbb{T}\mathrm{I} : \mu,x) \in \mathcal{F}_k$. By property 3 of Proposition 4.31, $\langle \mathcal{F}_k, \mathcal{C}_k\rangle \in \mathcal{P}$. Then $\langle \mathcal{F}_{k+1}, \mathcal{C}_{k+1}\rangle = \langle \mathcal{F}_k, \mathcal{C}_k \cup \{x \simeq \varepsilon\}\rangle$. Thus $x \simeq \varepsilon \in \overline{\mathcal{C}_\infty}$. Therefore the condition 6 of Definition 4.21 holds.
7 This case is similar to condition 14.
8 This case is similar to condition 14.
9 This case is similar to condition 14.
10 This case is similar to condition 14.
11 This case is similar to condition 14.
12 This case is similar to condition 14.
13 We suppose that $(\mathbb{T}\varphi \to \psi : \mu,x) \in \mathcal{F}_\infty$. As $\mathcal{P}$ is saturated then $\langle \mathcal{F}_\infty \cup \{(\mathbb{F}\varphi : \mu,x)\}, \mathcal{C}_\infty\rangle \in \mathcal{P}$ or $\langle \mathcal{F}_\infty \cup \{(\mathbb{T}\psi : \mu,x)\}, \mathcal{C}_\infty\rangle \in \mathcal{P}$, by rule $\langle \mathbb{T} \to\rangle$. By property 2 of Proposition 4.32, $(\mathbb{F}\varphi : \mu,x) \in \mathcal{F}_\infty$ or $(\mathbb{T}\psi : \mu,x) \in \mathcal{F}_\infty$. Therefore the condition 13 of Definition 4.21 holds.
14 We suppose that $(\mathbb{F}\varphi \to \psi : \mu,x) \in \mathcal{F}_\infty$. As $\mathcal{P}$ is saturated then $\langle \mathcal{F}_\infty \cup \{(\mathbb{T}\varphi : \mu,x), (\mathbb{F}\psi :$

$\mu,x)\}, \mathcal{C}_\infty\rangle \in \mathcal{P}$ by rule $\langle \mathbb{F} \to\rangle$. As $\mathcal{P}$ is $\preccurlyeq$-closed then $\langle \mathcal{F}_\infty \cup \{(\mathbb{T}\varphi : \mu,x)\}, \mathcal{C}_\infty\rangle \in \mathcal{P}$ and $\langle \mathcal{F}_\infty \cup \{(\mathbb{F}\psi : \mu,x)\}, \mathcal{C}_\infty\rangle \in \mathcal{P}$. By property 2 of Proposition 4.32, $(\mathbb{T}\varphi : \mu,x) \in \mathcal{F}_\infty$ and $(\mathbb{F}\psi : \mu,x) \in \mathcal{F}_\infty$. Therefore the condition 14 of Definition 4.21 holds.

15 We suppose that $(\mathbb{T}\varphi * \psi : \mu,x) \in \mathcal{F}_\infty$. By same arguments to that of condition 6, there is $k \in \mathbb{N}$ such that:

— $\mathcal{S}_k = (\mathbb{T}\varphi * \psi : \mu,x)$

— $(\mathbb{T}\varphi * \psi : \mu,x) \in \mathcal{F}_k$

— $\langle \mathcal{F}_k, \mathcal{C}_k\rangle \in \mathcal{P}$.

Then, by construction of the limit CSS, $\langle \mathcal{F}_{k+1}, \mathcal{C}_{k+1}\rangle = \langle \mathcal{F}_k \cup \{(\mathbb{T}\varphi : \mu, \mathfrak{a}), (\mathbb{T}\psi : \mu, \mathfrak{b})\}, \mathcal{C}_k \cup \{x \simeq \mathfrak{a}\mathfrak{b}\}\rangle$, where $\mathfrak{a} = c_{2k+2}$ and $\mathfrak{b} = c_{2k+3}$. Then $x \simeq \mathfrak{a}\mathfrak{b} \in \overline{\mathcal{C}_\infty}$, $(\mathbb{T}\varphi : \mu, \mathfrak{a}) \in \mathcal{F}_\infty$ and $(\mathbb{T}\psi : \mu, \mathfrak{b}) \in \mathcal{F}_\infty$. Therefore the condition 15 of Definition 4.21 holds.

16 We suppose that $(\mathbb{F}\varphi * \psi : \mu,x) \in \mathcal{F}_\infty$. Let $y,z \in L_r$ such that $x \simeq yz \in \overline{\mathcal{C}_\infty}$. As $\mathcal{P}$ is saturated then we have $\langle \mathcal{F}_\infty \cup \{(\mathbb{F}\varphi : \mu,y)\}, \mathcal{C}_\infty\rangle \in \mathcal{P}$ or $\langle \mathcal{F}_\infty \cup \{(\mathbb{F}\psi : \mu,z)\}, \mathcal{C}_\infty\rangle \in \mathcal{P}$, by rule $\langle \mathbb{F}*\rangle$. By property 2 of Proposition 4.32, $(\mathbb{F}\varphi : \mu,y) \in \mathcal{F}_\infty$ or $(\mathbb{F}\psi : \mu,z) \in \mathcal{F}_\infty$. Therefore the condition 16 of Definition 4.21 holds.

17 Similar to condition 16.

18 Similar to condition 15.

19 We suppose that $(\mathbb{T}K_u\varphi : \mu,x) \in \mathcal{F}_\infty$. Let $y \in L_r$ such that $x \sim_u^\mu y \in \overline{\mathcal{C}_\infty}$. As $\mathcal{P}$ is saturated then $\langle \mathcal{F}_\infty \cup \{(\mathbb{T}\varphi : \mu,y)\}, \mathcal{C}_\infty\rangle \in \mathcal{P}$, by the rule $\langle \mathbb{T}K_u\rangle$. By property 2 of Proposition 4.32, $(\mathbb{T}\varphi : \mu,y) \in \mathcal{F}_\infty$. Therefore the condition 19 of Definition 4.21 holds.

20 We suppose that $(\mathbb{F}K_u\varphi : \mu,x) \in \mathcal{F}_\infty$. By same arguments to that of condition 6, there is $k \in \mathbb{N}$ such that:

— $\mathcal{S}_k = (\mathbb{F}K_u\varphi : \mu,x)$

— $(\mathbb{F}K_u\varphi : \mu,x) \in \mathcal{F}_k$

— $\langle \mathcal{F}_k, \mathcal{C}_k\rangle \in \mathcal{P}$.

Then, by construction of the limit CSS, $\langle \mathcal{F}_{k+1}, \mathcal{C}_{k+1}\rangle = \langle \mathcal{F}_k \cup \{(\mathbb{F}\varphi : \mu, \mathfrak{a})\}, \mathcal{C}_k \cup \{x \sim_u^\mu \mathfrak{a}\}\rangle$, where $\mathfrak{a} = c_{2k+2}$. Then $x \sim_u^\mu \mathfrak{a} \in \overline{\mathcal{C}_\infty}$ and $(\mathbb{F}\varphi : \mu, \mathfrak{a}) \in \mathcal{F}_\infty$. Therefore the condition 20 of Definition 4.21 holds.

21 Similar to condition 20.

22 Similar to condition 19.

23 We suppose that $(\mathbb{T}[\varphi]\psi : \mu,x) \in \mathcal{F}_\infty$. As $\mathcal{P}$ is saturated then $\langle \mathcal{F}_\infty \cup \{(\mathbb{F}\varphi : \mu,x)\}, \mathcal{C}_\infty\rangle \in \mathcal{P}$ or $\langle \mathcal{F}_\infty \cup \{(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!],x)\}, \mathcal{C}_\infty \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle \in \mathcal{P}$, by the rule $\langle \mathbb{T}[\cdot]\rangle$. As $\mathcal{P}$ is $\preccurlyeq$-closed then $\langle \mathcal{F}_\infty \cup \{(\mathbb{F}\varphi : \mu,x)\}, \mathcal{C}_\infty\rangle \in \mathcal{P}$ or $\langle \mathcal{F}_\infty \cup \{(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!],x)\}, \mathcal{C}_\infty\rangle \in \mathcal{P}$. By Proposition 4.32, $(\mathbb{F}\varphi : \mu,y) \in \mathcal{F}_\infty$ or $(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!],x) \in \mathcal{F}_\infty$. Therefore the condition 23 of Definition 4.21 holds.

24 We suppose that $(\mathbb{F}[\varphi]\psi : \mu,x) \in \mathcal{F}_\infty$. As $\mathcal{P}$ is saturated then $\langle \mathcal{F}_\infty \cup \{(\mathbb{T}\varphi : \mu,x), (\mathbb{F}\psi : \mu \oplus [\![\varphi]\!],x)\}, \mathcal{C}_\infty \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle \in \mathcal{P}$, by the rule $\langle \mathbb{T}[\cdot]\rangle$. As $\mathcal{P}$ is $\preccurlyeq$-closed then $\langle \mathcal{F}_\infty \cup \{(\mathbb{T}\varphi : \mu,x)\}, \mathcal{C}_\infty\rangle \in \mathcal{P}$ and $\langle \mathcal{F}_\infty \cup \{(\mathbb{F}\psi : \mu \oplus [\![\varphi]\!],x)\}, \mathcal{C}_\infty\rangle \in \mathcal{P}$. By Proposition 4.32, $(\mathbb{T}\varphi : \mu,y) \in \mathcal{F}_\infty$ and $(\mathbb{F}\psi : \mu \oplus [\![\varphi]\!],x) \in \mathcal{F}_\infty$. Therefore the condition 24 of Definition 4.21 holds.

25 Similar to condition 24.

26 Similar to condition 23.

27 We suppose that $x \sim_u^{[\![\psi_1;\ldots;\psi_k]\!]} y \in \overline{\mathcal{C}_\infty}$.

As $\mathcal{P}$ is saturated then $\langle \mathcal{F}_\infty \cup \{(\mathbb{T}\psi_k : [\![\psi_1;...;\psi_{k-1}]\!], x), (\mathbb{T}\psi_k : [\![\psi_1;...;\psi_{k-1}]\!], y)\}, \mathcal{C}_\infty\rangle \in \mathcal{P}$ or $\langle \mathcal{F}_\infty \cup \{(\mathbb{F}\psi_k : [\![\psi_1;...;\psi_{k-1}]\!], x), (\mathbb{F}\psi_k : [\![\psi_1;...;\psi_{k-1}]\!], y)\}, \mathcal{C}_\infty\rangle \in \mathcal{P}$, by the rule $\langle R_{pop}\rangle$. As $\mathcal{P}$ is $\preccurlyeq$-closed, and by Proposition 4.32, we have $(\mathbb{T}\psi_k : [\![\psi_1;...;\psi_{k-1}]\!], x) \in \mathcal{F}_\infty$ and $(\mathbb{T}\psi_k : [\![\psi_1;...;\psi_{k-1}]\!], y) \in \mathcal{F}_\infty$, or $(\mathbb{F}\psi_k : [\![\psi_1;...;\psi_{k-1}]\!], x) \in \mathcal{F}_\infty$ and $(\mathbb{F}\psi_k : [\![\psi_1;...;\psi_{k-1}]\!], y) \in \mathcal{F}_\infty$.

28 We suppose that $x \eqcirc_u^\mu y \in \overline{\mathcal{C}_\infty}$ and $\triangleright\mu \oplus [\![\varphi]\!] \in \overline{\mathcal{C}_\infty}$. As $\mathcal{P}$ is saturated then one of the following conditions hold, by the rule $\langle R_{push}\rangle$:

— $\langle \mathcal{F}_\infty, \mathcal{C}_\infty \cup \{x \eqcirc_u^{\mu\oplus[\![\varphi]\!]} y\}\rangle \in \mathcal{P}$. In this case, by Proposition 4.32, we have $x \eqcirc_u^{\mu\oplus[\![\varphi]\!]} y \in \mathcal{C}_\infty$.

— $\langle \mathcal{F}_\infty \cup \{(\mathbb{T}\varphi : \mu, x), (\mathbb{F}\varphi : \mu, y)\}, \mathcal{C}_\infty\rangle \in \mathcal{P}$. In this case, as $\mathcal{P}$ is $\preccurlyeq$-closed and by Proposition 4.32, we have $(\mathbb{T}\varphi : \mu, x) \in \mathcal{F}_\infty$ and $(\mathbb{F}\varphi : \mu, y) \in \mathcal{F}_\infty$.

— $\langle \mathcal{F}_\infty \cup \{(\mathbb{F}\varphi : \mu, x), (\mathbb{T}\varphi : \mu, y)\}, \mathcal{C}_\infty\rangle \in \mathcal{P}$. In this case, as $\mathcal{P}$ is $\preccurlyeq$-closed and by Proposition 4.32, we have $(\mathbb{F}\varphi : \mu, x) \in \mathcal{F}_\infty$ and $(\mathbb{T}\varphi : \mu, y) \in \mathcal{F}_\infty$.

□

Appendix G. Proof of Lemma 4.30

Definition G.1 (Consistency property). A *consistency property* is a set $\mathcal{P}$ of CSS satisfying the following conditions for all CSS $\langle \mathcal{F}, \mathcal{C}\rangle \in \mathcal{P}$, all $\varphi, \psi \in \mathcal{L}$, all $u \in A$, all $\mu \in \mathfrak{L}_\mathcal{L}$ and all $x, y \in L_r$:

1 $(\mathbb{T}p : \mu, x) \notin \mathcal{F}$ or $(\mathbb{F}p : \kappa, y) \notin \mathcal{F}$ or $x \simeq y \notin \overline{\mathcal{C}}$

2 $(\mathbb{T}\varphi : \mu, x) \notin \mathcal{F}$ or $(\mathbb{F}\varphi : \mu, y) \notin \mathcal{F}$ or $x \simeq y \notin \overline{\mathcal{C}}$

3 $(\mathbb{F}\mathrm{I} : \mu, x) \notin \mathcal{F}$ or $x \simeq \varepsilon \notin \overline{\mathcal{C}}$

4 $(\mathbb{F}\top : \mu, x) \notin \mathcal{F}$

5 $(\mathbb{T}\bot : \mu, x) \notin \mathcal{F}$

6 If $(\mathbb{T}\mathrm{I} : \mu, x) \in \mathcal{F}$ then $\langle \mathcal{F}, \mathcal{C} \cup \{x \simeq \varepsilon\}\rangle \in \mathcal{P}$

7 If $(\mathbb{T}\neg\varphi : \mu, x) \in \mathcal{F}$ then $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, x)\}, \mathcal{C}\rangle \in \mathcal{P}$

8 If $(\mathbb{F}\neg\varphi : \mu, x) \in \mathcal{F}$ then $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, x)\}, \mathcal{C}\rangle \in \mathcal{P}$

9 If $(\mathbb{T}\varphi \wedge \psi : \mu, x) \in \mathcal{F}$ then $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, x), (\mathbb{T}\psi : \mu, x)\}, \mathcal{C}\rangle \in \mathcal{P}$

10 If $(\mathbb{F}\varphi \wedge \psi : \mu, x) \in \mathcal{F}$ then $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, x)\}, \mathcal{C}\rangle \in \mathcal{P}$ or $\langle \mathcal{F} \cup \{(\mathbb{F}\psi : \mu, x)\}, \mathcal{C}\rangle \in \mathcal{P}$

11 If $(\mathbb{T}\varphi \vee \psi : \mu, x) \in \mathcal{F}$ then $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, x)\}, \mathcal{C}\rangle \in \mathcal{P}$ or $\langle \mathcal{F} \cup \{(\mathbb{T}\psi : \mu, x)\}, \mathcal{C}\rangle \in \mathcal{P}$

12 If $(\mathbb{F}\varphi \vee \psi : \mu, x) \in \mathcal{F}$ then $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, x), (\mathbb{F}\psi : \mu, x)\}, \mathcal{C}\rangle \in \mathcal{P}$

13 If $(\mathbb{T}\varphi \rightarrow \psi : \mu, x) \in \mathcal{F}$ then $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, x)\}, \mathcal{C}\rangle \in \mathcal{P}$ or $\langle \mathcal{F} \cup \{(\mathbb{T}\psi : \mu, x)\}, \mathcal{C}\rangle \in \mathcal{P}$

14 If $(\mathbb{F}\varphi \rightarrow \psi : \mu, x) \in \mathcal{F}$ then $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, x), (\mathbb{F}\psi : \mu, x)\}, \mathcal{C}\rangle \in \mathcal{P}$

15 If $(\mathbb{T}\varphi * \psi : \mu, x) \in \mathcal{F}$ then $\exists c_i, c_j \in \gamma_r \setminus \mathcal{A}_r(\mathcal{C}), c_i \neq c_j$ and $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, c_i), (\mathbb{T}\psi : \mu, c_j)\}, \mathcal{C} \cup \{x \simeq c_i c_j\}\rangle \in \mathcal{P}$

16 If $(\mathbb{F}\varphi * \psi : \mu, x) \in \mathcal{F}$ then $\forall y, z \in L_r, x \simeq yz \in \overline{\mathcal{C}} \Rightarrow \langle \mathcal{F} \cup \{(\mathbb{F}\mu : \varphi, y)\}, \mathcal{C}\rangle \in \mathcal{P}$ or $\langle \mathcal{F} \cup \{(\mathbb{F}\psi : \mu, z)\}, \mathcal{C}\rangle \in \mathcal{P}$

17 If $(\mathbb{T}\varphi \mathrel{-\!\!*} \psi : \mu, x) \in \mathcal{F}$ then $\forall y \in L_r, xy \simeq xy \in \overline{\mathcal{C}} \Rightarrow \langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, y)\}, \mathcal{C}\rangle \in \mathcal{P}$ or $\langle \mathcal{F} \cup \{(\mathbb{T}\psi : \mu, xy)\}, \mathcal{C}\rangle \in \mathcal{P}$

18 If $(\mathbb{F}\varphi \mathrel{-\!\!*} \psi : \mu, x) \in \mathcal{F}$ then $\exists c_i \in \gamma_r \setminus \mathcal{A}_r(\mathcal{C})$, $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, c_i), (\mathbb{F}\psi : \mu, xc_i)\}, \mathcal{C} \cup \{xc_i \simeq xc_i\}\rangle \in \mathcal{P}$

19 If $(\mathbb{T}K_u\varphi : \mu, x) \in \mathcal{F}$ then $\forall y \in L_r, x \eqcirc_u^\mu y \in \overline{\mathcal{C}} \Rightarrow \langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, y)\}, \mathcal{C}\rangle \in \mathcal{P}$

20 If $(\mathbb{F}K_u\varphi : \mu, x) \in \mathcal{F}$ then $\exists c_i \in \gamma_r \setminus \mathcal{A}_r(\mathcal{C})$, $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, c_i)\}, \mathcal{C} \cup \{x \eqcirc_u^\mu c_i\}\rangle \in \mathcal{P}$

21 If $(\mathbb{T}\widetilde{K}_u\varphi : \mu, x) \in \mathcal{F}$ then $\exists c_i \in \gamma_r \setminus \mathcal{A}_r(\mathcal{C})$, $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, c_i)\}, \mathcal{C} \cup \{x \eqcirc_u^\mu c_i\}\rangle \in \mathcal{P}$

22 If $(\mathbb{F}\widetilde{K}_u\varphi : \mu, x) \in \mathcal{F}$ then $\forall y \in L_r, x \eqcirc_u^\mu y \in \overline{\mathcal{C}} \Rightarrow \langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, y)\}, \mathcal{C}\rangle \in \mathcal{P}$

23 If $(\mathbb{T}[\varphi]\psi : \mu, x) \in \mathcal{F}$ then $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, x)\}, \mathcal{C}\rangle \in \mathcal{P}$ or $\langle \mathcal{F} \cup \{(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!], x)\}, \mathcal{C} \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle \in \mathcal{P}$
24 If $(\mathbb{F}[\varphi]\psi : \mu, x) \in \mathcal{F}$ then $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, x), (\mathbb{F}\psi : \mu \oplus [\![\varphi]\!], x)\}, \mathcal{C} \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle \in \mathcal{P}$
25 If $(\mathbb{T}\langle\varphi\rangle\psi : \mu, x) \in \mathcal{F}$ then $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, x), (\mathbb{T}\psi : \mu \oplus [\![\varphi]\!], x)\}, \mathcal{C} \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle \in \mathcal{P}$
26 If $(\mathbb{F}\langle\varphi\rangle\psi : \mu, x) \in \mathcal{F}$ then $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, x)\}, \mathcal{C}\rangle \in \mathcal{P}$ or $\langle \mathcal{F} \cup \{(\mathbb{F}\psi : \mu \oplus [\![\varphi]\!], x)\}, \mathcal{C} \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle \in \mathcal{P}$
27 If $x \sim_u^{[\![\psi_1;\ldots;\psi_k]\!]} y \in \overline{\mathcal{C}}$ then one of the following conditions hold:
- $\langle \mathcal{F} \cup \{(\mathbb{T}\psi_k : [\![\psi_1;\ldots;\psi_{k-1}]\!], x), (\mathbb{T}\psi_k : [\![\psi_1;\ldots;\psi_{k-1}]\!], y)\}, \mathcal{C}\rangle \in \mathcal{P}$
- $\langle \mathcal{F} \cup \{(\mathbb{F}\psi_k : [\![\psi_1;\ldots;\psi_{k-1}]\!], x), (\mathbb{F}\psi_k : [\![\psi_1;\ldots;\psi_{k-1}]\!], y)\}, \mathcal{C}\rangle \in \mathcal{P}$

28 If $x \sim_u^{\mu} y \in \overline{\mathcal{C}}$ and $\triangleright\mu \oplus [\![\varphi]\!] \in \overline{\mathcal{C}}$ then one of the following conditions hold:
- $\langle \mathcal{F}, \mathcal{C} \cup \{x \sim_u^{\mu\oplus[\![\varphi]\!]} y\}\rangle \in \mathcal{P}$
- $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, x), (\mathbb{F}\varphi : \mu, y)\}, \mathcal{C}\rangle \in \mathcal{P}$
- $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, x), (\mathbb{T}\varphi : \mu, y)\}, \mathcal{C}\rangle \in \mathcal{P}$

Conditions 1, 2, 3, 4 and 5 correspond to conditions that ensure that the CSS are not closed. Other conditions ensure that if we apply a rule of Fig. 7 then one of the new CSS belongs to $\mathcal{P}$. We can remark that all new CSS satisfy the condition (P_{css}) of Definition 4.12.

Definition G.2 (Alternate consistency property). An *alternate consistency property* is a set $\mathcal{P}$ of CSS satisfying the conditions of consistency property, except conditions 15, 18, 20 and 21 which are respectively replaced by 15', 18', 20' and 21'.

15'.If $(\mathbb{T}\varphi * \psi : \mu, x) \in \mathcal{F}$ then $\forall c_i \neq c_j \in \gamma_r \setminus \mathcal{A}_r(\mathcal{C})$, $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, c_i), (\mathbb{T}\psi : \mu, c_j)\}, \mathcal{C} \cup \{x \simeq c_i c_j\}\rangle \in \mathcal{P}$
18'.If $(\mathbb{F}\varphi -\!\!* \psi : \mu, x) \in \mathcal{F}$ then $\forall c_i \in \gamma_r \setminus \mathcal{A}_r(\mathcal{C})$, $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, c_i), (\mathbb{F}\psi : \mu, xc_i)\}, \mathcal{C} \cup \{xc_i \simeq xc_i\}\rangle \in \mathcal{P}$
20'.If $(\mathbb{F}K_u\varphi : \mu, x) \in \mathcal{F}$ then $\forall c_i \in \gamma_r \setminus \mathcal{A}_r(\mathcal{C})$, $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, c_i)\}, \mathcal{C} \cup \{x \sim_u^{\mu} c_i\}\rangle \in \mathcal{P}$
21'.If $(\mathbb{T}\widetilde{K}_u\varphi : \mu, x) \in \mathcal{F}$ then $\forall c_i \in \gamma_r \setminus \mathcal{A}_r(\mathcal{C})$, $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, c_i)\}, \mathcal{C} \cup \{x \sim_u^{\mu} c_i\}\rangle \in \mathcal{P}$

Proposition G.3. The set of every finite CSS for which there exists no closed tableau is a consistency property.

Proof. Let $\mathcal{P}$ the set of finite CSS for which there exists no closed tableau. We show that $\mathcal{P}$ is a consistency property. Let $\langle \mathcal{F}, \mathcal{C}\rangle \in \mathcal{P}$.

- If $(\mathbb{T}p : \mu, x) \in \mathcal{F}$ and $(\mathbb{F}p : \kappa, y) \in \mathcal{F}$ and $x \simeq y \in \overline{\mathcal{C}}$ then $\langle \mathcal{F}, \mathcal{C}\rangle$ is closed. But this is contradictory, because there is no closed tableau for this CSS.
- If $(\mathbb{T}K_u\varphi : \mu, x) \in \mathcal{F}$. Let $y \in L_r$ such that $x \sim_u^{\mu} y \in \overline{\mathcal{C}}$. We suppose that $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, y)\}, \mathcal{C}\rangle \notin \mathcal{P}$. By the rule $\langle \mathbb{T}K_u\rangle$, $[\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, y)\}, \mathcal{C}\rangle]$ is a finite tableau for $\langle \mathcal{F}, \mathcal{C}\rangle$. Then $\langle \mathcal{F}, \mathcal{C}\rangle$ has a closed tableau, which is contradictory. Hence $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, y)\}, \mathcal{C}\rangle \in \mathcal{P}$.
- If $(\mathbb{F}K_u\varphi : \mu, x) \in \mathcal{F}$. We choose $c_i \in \gamma_r \setminus \mathcal{A}_r(\mathcal{C})$ (remember that γ_r is infinite and $\mathcal{A}_r(\mathcal{C})$ is finite because $\mathcal{C}$ is finite, by hypothesis). We suppose that $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, c_i)\}, \mathcal{C} \cup \{x \sim_u^{\mu} c_i\}\rangle \notin \mathcal{P}$. By the rule $\langle \mathbb{F}K_u\rangle$, $[\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, c_i)\}, \mathcal{C} \cup \{x \sim_u^{\mu} c_i\}\rangle]$ is a finite tableau for $\langle \mathcal{F}, \mathcal{C}\rangle$. Thus $\langle \mathcal{F}, \mathcal{C}\rangle$ has a closed tableau, which is contradictory. Hence $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, c_i)\}, \mathcal{C} \cup \{x \sim_u^{\mu} c_i\}\rangle \in \mathcal{P}$.

— If $(\mathbb{T}[\varphi]\psi : \mu,x) \in \mathcal{F}$. We suppose that $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu,x)\}, \mathcal{C}\rangle \notin \mathcal{P}$ and $\langle \mathcal{F} \cup \{(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!],x)\}, \mathcal{C} \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle \notin \mathcal{P}$. By the rule $\langle \mathbb{T}[\cdot]\rangle$, $[\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu,x)\}, \mathcal{C}\rangle; \langle \mathcal{F} \cup \{(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!],x)\}, \mathcal{C} \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle]$ is a finite tableau for $\langle \mathcal{F}, \mathcal{C}\rangle$. Thus $\langle \mathcal{F}, \mathcal{C}\rangle$ has a closed tableau, which is contradictory. Hence $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu,x)\}, \mathcal{C}\rangle \in \mathcal{P}$ or $\langle \mathcal{F} \cup \{(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!],x)\}, \mathcal{C} \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle \in \mathcal{P}$.

— If $(\mathbb{F}[\varphi]\psi : \mu,x) \in \mathcal{F}$. We suppose that $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu,x), (\mathbb{F}\psi : \mu \oplus [\![\varphi]\!],x)\}, \mathcal{C} \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle \notin \mathcal{P}$. By the rule $\langle \mathbb{F}[\cdot]\rangle$, $[\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu,x), (\mathbb{F}\psi : \mu \oplus [\![\varphi]\!],x)\}, \mathcal{C} \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle]$ is a finite tableau for $\langle \mathcal{F}, \mathcal{C}\rangle$. Thus $\langle \mathcal{F}, \mathcal{C}\rangle$ has a closed tableau, which is contradictory. Hence $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu,x), (\mathbb{F}\psi : \mu \oplus [\![\varphi]\!],x)\}, \mathcal{C} \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle \in \mathcal{P}$.

— Other cases are proved similarly.

In conclusion, $\mathcal{P}$ is a consistency property. □

Proposition G.4. Any consistency property can be extended into a $\preccurlyeq$-closed consistency property.

Proof. Let $\mathcal{P}$ be a consistency property. Let $\mathcal{P}^{\preccurlyeq}$ its $\preccurlyeq$-closure defined by:

$$\langle \mathcal{F}, \mathcal{C}\rangle \in \mathcal{P}^{\preccurlyeq} \text{ iff } \langle \mathcal{F}, \mathcal{C}\rangle \preccurlyeq \langle \mathcal{F}', \mathcal{C}'\rangle \text{ for some } \langle \mathcal{F}', \mathcal{C}'\rangle \in \mathcal{P}$$

We have $\mathcal{P} \subseteq \mathcal{P}^{\preccurlyeq}$ because $\preccurlyeq$ is reflexive. $\mathcal{P}^{\preccurlyeq}$ is $\preccurlyeq$-closed because $\preccurlyeq$ is transitive. We show now that $\mathcal{P}^{\preccurlyeq}$ is a consistency property.
Let $\langle \mathcal{F}, \mathcal{C}\rangle \in \mathcal{P}^{\preccurlyeq}$. Then there exists $\langle \mathcal{F}', \mathcal{C}'\rangle \in \mathcal{P}$ such that $\langle \mathcal{F}, \mathcal{C}\rangle \preccurlyeq \langle \mathcal{F}', \mathcal{C}'\rangle$.

— We suppose that $(\mathbb{T}p : \mu,x) \in \mathcal{F}$ and $(\mathbb{F}p : \kappa,y) \in \mathcal{F}$ and $x \simeq y \in \overline{\mathcal{C}}$. Then $(\mathbb{T}p : \mu,x) \in \mathcal{F}'$ and $(\mathbb{F}p : \kappa,y) \in \mathcal{F}'$ and $x \simeq y \in \overline{\mathcal{C}'}$ because $\langle \mathcal{F}, \mathcal{C}\rangle \preccurlyeq \langle \mathcal{F}', \mathcal{C}'\rangle$. But this is contradictory because $\langle \mathcal{F}', \mathcal{C}'\rangle \in \mathcal{P}$ and $\mathcal{P}$ satisfies condition 1 of Definition **??**.

— We suppose that $(\mathbb{T}K_u\varphi : \mu,x) \in \mathcal{F}$. Let $y \in L_r$ such that $x =^{\mu}_{u} y \in \overline{\mathcal{C}}$. As $\langle \mathcal{F}, \mathcal{C}\rangle \preccurlyeq \langle \mathcal{F}', \mathcal{C}'\rangle$ then $(\mathbb{T}K_u\varphi : \mu,x) \in \mathcal{F}'$ and $x =^{\mu}_{u} y \in \overline{\mathcal{C}'}$. Thus $\langle \mathcal{F}' \cup \{(\mathbb{T}\varphi : \mu,y)\}, \mathcal{C}'\rangle \in \mathcal{P}$. Moreover $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu,y)\}, \mathcal{C}\rangle \preccurlyeq \langle \mathcal{F}' \cup \{(\mathbb{T}\varphi : \mu,y)\}, \mathcal{C}'\rangle$. Hence $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu,y)\}, \mathcal{C}\rangle \in \mathcal{P}^{\preccurlyeq}$.

— We suppose that $(\mathbb{F}K_u\varphi : \mu,x) \in \mathcal{F}$. As $\langle \mathcal{F}, \mathcal{C}\rangle \preccurlyeq \langle \mathcal{F}', \mathcal{C}'\rangle$ then $(\mathbb{F}K_u\varphi : \mu,x) \in \mathcal{F}'$. Thus there is $c_i \in \gamma_r \setminus \mathcal{A}_r(\mathcal{C}')$ such that $\langle \mathcal{F}' \cup \{(\mathbb{F}\varphi : \mu,c_i)\}, \mathcal{C}' \cup \{x =^{\mu}_{u} c_i\}\rangle \in \mathcal{P}$. As $\mathcal{C} \subseteq \mathcal{C}'$ then $\mathcal{A}_r(\mathcal{C}) \subseteq \mathcal{A}_r(\mathcal{C}')$. Then $\gamma_r \setminus \mathcal{A}_r(\mathcal{C}') \subseteq \gamma_r \setminus \mathcal{A}_r(\mathcal{C})$. Hence $c_i \in \gamma_r \setminus \mathcal{A}_r(\mathcal{C})$. Moreover $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu,c_i)\}, \mathcal{C} \cup \{x =^{\mu}_{u} c_i\}\rangle \preccurlyeq \langle \mathcal{F}' \cup \{(\mathbb{F}\varphi : \mu,c_i)\}, \mathcal{C}' \cup \{x =^{\mu}_{u} c_i\}\rangle$. Therefore $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu,c_i)\}, \mathcal{C} \cup \{x =^{\mu}_{u} c_i\}\rangle \in \mathcal{P}^{\preccurlyeq}$.

— We suppose that $(\mathbb{T}[\varphi]\psi : \mu,x) \in \mathcal{F}$. As $\langle \mathcal{F}, \mathcal{C}\rangle \preccurlyeq \langle \mathcal{F}', \mathcal{C}'\rangle$ then $(\mathbb{T}[\varphi]\psi : \mu,x) \in \mathcal{F}'$. Thus $\langle \mathcal{F}' \cup \{(\mathbb{F}\varphi : \mu,x)\}, \mathcal{C}'\rangle \in \mathcal{P}$ or $\langle \mathcal{F}' \cup \{(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!],x)\}, \mathcal{C}' \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle \in \mathcal{P}$. Moreover $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu,x)\}, \mathcal{C}\rangle \preccurlyeq \langle \mathcal{F}' \cup \{(\mathbb{F}\varphi : \mu,x), \mathcal{C}'\rangle$ and $\langle \mathcal{F} \cup \{(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!],x)\}, \mathcal{C} \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle \preccurlyeq \langle \mathcal{F}' \cup \{(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!],x)\}, \mathcal{C}' \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle$. Therefore $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu,x)\}, \mathcal{C}\rangle \in \mathcal{P}^{\preccurlyeq}$ or $\langle \mathcal{F} \cup \{(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!],x)\}, \mathcal{C} \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle \in \mathcal{P}^{\preccurlyeq}$.

— We suppose that $(\mathbb{F}[\varphi]\psi : \mu,x) \in \mathcal{F}$. As $\langle \mathcal{F}, \mathcal{C}\rangle \preccurlyeq \langle \mathcal{F}', \mathcal{C}'\rangle$ then $(\mathbb{F}[\varphi]\psi : \mu,x) \in \mathcal{F}'$. Thus $\langle \mathcal{F}' \cup \{(\mathbb{T}\varphi : \mu,x), (\mathbb{F}\psi : \mu \oplus [\![\varphi]\!],x)\}, \mathcal{C}' \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle \in \mathcal{P}$. Moreover $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu,x), (\mathbb{F}\psi : \mu \oplus [\![\varphi]\!],x)\}, \mathcal{C} \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle \preccurlyeq \langle \mathcal{F}' \cup \{(\mathbb{T}\varphi : \mu,x), (\mathbb{F}\psi : \mu \oplus [\![\varphi]\!],x)\}, \mathcal{C}' \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle$. Therefore $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu,x), (\mathbb{F}\psi : \mu \oplus [\![\varphi]\!],x)\}, \mathcal{C} \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle \in \mathcal{P}^{\preccurlyeq}$.

— Other cases are proved similarly.

□

Definition G.5 (Substitution). A *substitution* is a function $\sigma : \gamma_r \cup \{\varepsilon\} \longrightarrow \gamma_r \cup \{\varepsilon\}$, such that $\sigma(\varepsilon) = \varepsilon$.

We extend this definition to resource labels as follows: if $c_{i_1}, \ldots, c_{i_k} \in \gamma_r$ then $\sigma(c_{i_1} \ldots c_{i_k}) = \sigma(c_{i_1}) \ldots \sigma(c_{i_k})$.
We extend it to labelled formulae and constraints as follows:

— For any labelled formula $(\mathbb{S}\varphi : \mu, x)$, $\sigma(\mathbb{S}\varphi : \mu, x) = (\mathbb{S}\varphi : \mu, \sigma(x))$
— For any resource constraint $x \simeq y$, $\sigma(x \simeq y) = \sigma(x) \simeq \sigma(y)$
— For any agent constraint $x \doteq^{\mu}_{u} y$, $\sigma(x \doteq^{\mu}_{u} y) = \sigma(x) \doteq^{\mu}_{u} \sigma(y)$
— For any announcement constraint $\triangleright\mu$, $\sigma(\triangleright\mu) = \triangleright\mu$

We extend it to labelled formulae sets and constraint sets as follows:

— For any set of labelled formulae $\mathcal{F}$, $\sigma(\mathcal{F}) = \{(\mathbb{S}\varphi : \mu, \sigma(x)) \mid (\mathbb{S}\varphi : \mu, x) \in \mathcal{F}\}$
— For any set of constraints $\mathcal{C}$, $\sigma(\mathcal{C}) = \{\sigma(x) \simeq \sigma(y) \mid x \simeq y \in \mathcal{C}\} \cup \{\sigma(x) \doteq^{\mu}_{u} \sigma(y) \mid x \doteq^{\mu}_{u} y \in \mathcal{C}\} \cup \{\triangleright\mu \mid \triangleright\mu \in \mathcal{C}\}$

Lemma G.6. Let σ be a substitution and let $\mathcal{C}$ be a set of constraints. We have $\sigma(\overline{\mathcal{C}}) \subseteq \overline{\sigma(\mathcal{C})}$

Proof. Let $\mathcal{C}^\dagger$ defined by:

— $x \simeq y \in \mathcal{C}^\dagger$ iff $\sigma(x) \simeq \sigma(y) \in \overline{\sigma(\mathcal{C})}$
— $x \doteq^{\mu}_{u} y \in \mathcal{C}^\dagger$ iff $\sigma(x) \doteq^{\mu}_{u} \sigma(y) \in \overline{\sigma(\mathcal{C})}$
— $\triangleright\mu \in \mathcal{C}^\dagger$ iff $\triangleright\mu \in \overline{\sigma(\mathcal{C})}$

We show that $\overline{\mathcal{C}^\dagger} \subseteq \mathcal{C}^\dagger$. Let $c \in \overline{\mathcal{C}^\dagger}$ a constraint. We show that $c \in \mathcal{C}^\dagger$. There are two cases. The first case is: $c \in \overline{\mathcal{C}^\dagger}$ because $c \in \mathcal{C}^\dagger$: in this case, we have obviously $c \in \mathcal{C}^\dagger$. The second case is: c is obtained by rules of Fig. 6. We prove that $c \in \mathcal{C}^\dagger$ by induction on the size n of the deduction tree of c.

— Base case ($n = 0$):
In this case, the deduction tree is of the form $\dfrac{}{\varepsilon \simeq \varepsilon}\langle\varepsilon\rangle$ and c is the constraint $\varepsilon \simeq \varepsilon$. By rule $\langle\varepsilon\rangle$, $\varepsilon \simeq \varepsilon \in \overline{\sigma(\mathcal{C})}$. As $\sigma(\varepsilon) = \varepsilon$ then $\sigma(\varepsilon) \simeq \sigma(\varepsilon) \in \overline{\sigma(\mathcal{C})}$. By definition of $\mathcal{C}^\dagger$, $\varepsilon \simeq \varepsilon \in \mathcal{C}^\dagger$. Then, $c \in \mathcal{C}^\dagger$.
— Inductive step:
We suppose that any constraint of $\overline{\mathcal{C}^\dagger}$ having a deduction tree such that its size is lower or equal to n (IH) belongs to $\mathcal{C}^\dagger$. We show that $c \in \mathcal{C}^\dagger$ for any constraint $c \in \overline{\mathcal{C}^\dagger}$ such that the deduction tree size of c is equal to $n+1$.
  – Case $\langle c_r\rangle$:
  The deduction tree is of the form $\dfrac{\dfrac{\vdots}{x \simeq y} \quad \dfrac{\vdots}{yk \simeq yk}}{xk \simeq yk}\langle c_r\rangle$ and c is the constraint $xk \simeq yk$. By (IH), we have $x \simeq y \in \mathcal{C}^\dagger$ and $yk \simeq yk \in \mathcal{C}^\dagger$. By definition, $\sigma(x) \simeq \sigma(y) \in \overline{\sigma(\mathcal{C})}$ and $\sigma(yk) \simeq \sigma(yk) \in \overline{\sigma(\mathcal{C})}$. By definition, $\sigma(y)\sigma(k) \simeq \sigma(y)\sigma(k) \in \overline{\sigma(\mathcal{C})}$. By rule $\langle c_r\rangle$, $\sigma(x)\sigma(k) \simeq \sigma(y)\sigma(k) \in \overline{\sigma(\mathcal{C})}$. Thus $\sigma(xk) \simeq \sigma(yk) \in \overline{\sigma(\mathcal{C})}$. Hence $xk \simeq yk \in \mathcal{C}^\dagger$.
  – Case $\langle k_r\rangle$:
  The deduction tree is of the form $\dfrac{\dfrac{\vdots}{x \doteq^{\mu}_{u} y}}{x \simeq x}\langle k_r\rangle$ and c is the constraint $x \simeq x$. By (IH), we

have $x \mathrel{\dot{=}}^{\mu}_{u} y \in \mathcal{C}^{\dagger}$. By definition, $\sigma(x) \mathrel{\dot{=}}^{\mu}_{u} \sigma(y) \in \overline{\sigma(\mathcal{C})}$. By rule $\langle k_r \rangle$, $\sigma(x) \simeq \sigma(x) \in \overline{\sigma(\mathcal{C})}$. Hence $x \simeq x \in \mathcal{C}^{\dagger}$.

– Case $\langle a_n \rangle$:

The deduction tree is of the form $\dfrac{\dfrac{\vdots}{x \mathrel{\dot{=}}^{\mu}_{u} y}}{\rhd\mu}\langle a_n \rangle$ and c is the constraint $\rhd\mu$. By (IH), we have $x \mathrel{\dot{=}}^{\mu}_{u} y \in \mathcal{C}^{\dagger}$. By definition, $\sigma(x) \mathrel{\dot{=}}^{\mu}_{u} \sigma(y) \in \overline{\sigma(\mathcal{C})}$. By rule $\langle a_n \rangle$, $\rhd\mu \in \overline{\sigma(\mathcal{C})}$. Hence $\rhd\mu \in \mathcal{C}^{\dagger}$.

– Other cases are proved similarly.

Thus $\overline{\mathcal{C}^{\dagger}} \subseteq \mathcal{C}^{\dagger}$. By definition, $\sigma(\mathcal{C}) \subseteq \overline{\sigma(\mathcal{C})}$. Let $c \in \mathcal{C}$. We have $\sigma(c) \in \sigma(\mathcal{C})$. Then $\sigma(c) \in \overline{\sigma(\mathcal{C})}$ and $c \in \mathcal{C}^{\dagger}$, by definition of $\mathcal{C}^{\dagger}$. Therefore $\mathcal{C} \subseteq \mathcal{C}^{\dagger}$ and we have also $\overline{\mathcal{C}} \subseteq \overline{\mathcal{C}^{\dagger}}$. As $\overline{\mathcal{C}^{\dagger}} \subseteq \mathcal{C}^{\dagger}$, we have $\overline{\mathcal{C}} \subseteq \mathcal{C}^{\dagger}$. Now, let $c' \in \sigma(\overline{\mathcal{C}})$. There are three cases:

— Case $c' = x \simeq y \in \sigma(\overline{\mathcal{C}})$:
Then there is $m \simeq n \in \overline{\mathcal{C}}$ such that $x = \sigma(m)$ and $y = \sigma(n)$. As $\overline{\mathcal{C}} \subseteq \mathcal{C}^{\dagger}$ then $m \simeq n \in \mathcal{C}^{\dagger}$. Thus $\sigma(m) \simeq \sigma(n) \in \overline{\sigma(\mathcal{C})}$ and we have $c' \in \overline{\sigma(\mathcal{C})}$.

— Case $c' = x \mathrel{\dot{=}}^{\mu}_{u} y \in \sigma(\overline{\mathcal{C}})$:
Then there is $m \mathrel{\dot{=}}^{\mu}_{u} n \in \overline{\mathcal{C}}$ such that $x = \sigma(m)$ and $y = \sigma(n)$. As $\overline{\mathcal{C}} \subseteq \mathcal{C}^{\dagger}$ then $m \mathrel{\dot{=}}^{\mu}_{u} n \in \mathcal{C}^{\dagger}$. Thus $\sigma(m) \mathrel{\dot{=}}^{\mu}_{u} \sigma(n) \in \overline{\sigma(\mathcal{C})}$ and we have $c' \in \overline{\sigma(\mathcal{C})}$.

— Case $c' = \rhd\mu \in \sigma(\overline{\mathcal{C}})$:
Then $\rhd\mu \in \overline{\mathcal{C}}$. As $\overline{\mathcal{C}} \subseteq \mathcal{C}^{\dagger}$ then $\rhd\mu \in \mathcal{C}^{\dagger}$. Thus $\rhd\mu \in \overline{\sigma(\mathcal{C})}$ and we have $c' \in \overline{\sigma(\mathcal{C})}$.

Therefore $\sigma(\overline{\mathcal{C}}) \subseteq \overline{\sigma(\mathcal{C})}$. □

Corollary G.7. Let σ be a substitution and $\mathcal{C}$ be a set of constraints. If $c \in \overline{\mathcal{C}}$ then $\sigma(c) \in \overline{\sigma(\mathcal{C})}$.

Proof. Let $c \in \overline{\mathcal{C}}$. By definition, $\sigma(c) \in \sigma(\overline{\mathcal{C}})$. By Lemma G.6, we have $\sigma(\overline{\mathcal{C}}) \subseteq \overline{\sigma(\mathcal{C})}$. Thus $\sigma(c) \in \overline{\sigma(\mathcal{C})}$. □

Proposition G.8. Let σ be a substitution. The following properties hold:

1 If $\langle \mathcal{F}, \mathcal{C} \rangle$ is a CSS then $\langle \sigma(\mathcal{F}), \sigma(\mathcal{C}) \rangle$ is a CSS
2 If $\langle \mathcal{F}_f, \mathcal{C}_f \rangle$ is a finite CSS then $\langle \sigma(\mathcal{F}_f), \sigma(\mathcal{C}_f) \rangle$ is a finite CSS.
3 If $\langle \mathcal{F}, \mathcal{C} \rangle \preccurlyeq \langle \mathcal{F}', \mathcal{C}' \rangle$ then $\langle \sigma(\mathcal{F}), \sigma(\mathcal{C}) \rangle \preccurlyeq \langle \sigma(\mathcal{F}'), \sigma(\mathcal{C}') \rangle$.

Proof. Let σ be a substitution.

1 We suppose that $\langle \mathcal{F}, \mathcal{C} \rangle$ is a CSS. We have to show that $\langle \sigma(\mathcal{F}), \sigma(\mathcal{C}) \rangle$ satisfies the property (P_{css}) of Definition 4.12. Let $(\mathbb{S}\varphi : \mu, x) \in \sigma(\mathcal{F})$. By definition, there exists $(\mathbb{S}\varphi : \mu, x') \in \mathcal{F}$ such that $x = \sigma(x')$. As $\langle \mathcal{F}, \mathcal{C} \rangle$ is a CSS then, by (P_{css}) and Proposition 4.9, $x' \simeq x' \in \overline{\mathcal{C}}$ and $\rhd\mu \in \overline{\mathcal{C}}$. By Corollary G.7, $\sigma(x') \simeq \sigma(x') \in \overline{\sigma(\mathcal{C})}$ and $\rhd\mu \in \overline{\sigma(\mathcal{C})}$. Thus $x \simeq x \in \overline{\sigma(\mathcal{C})}$ and $\rhd\mu \in \overline{\sigma(\mathcal{C})}$. Therefore (P_{css}) holds and $\langle \sigma(\mathcal{F}), \sigma(\mathcal{C}) \rangle$ is a CSS.
2 Let $\langle \mathcal{F}_f, \mathcal{C}_f \rangle$ be a finite CSS. By previous property, $\langle \sigma(\mathcal{F}_f), \sigma(\mathcal{C}_f) \rangle$ is a CSS. We show now that $\langle \sigma(\mathcal{F}_f), \sigma(\mathcal{C}_f) \rangle$ is finite. As σ is surjective ans as $\mathcal{F}_f$ is finite then $\sigma(\mathcal{F}_f)$ is finite. The proof for $\sigma(\mathcal{C}_f)$ is similar. Thus $\langle \sigma(\mathcal{F}_f), \sigma(\mathcal{C}_f) \rangle$ is a finite CSS.
3 Let $\langle \mathcal{F}, \mathcal{C} \rangle$ and $\langle \mathcal{F}', \mathcal{C}' \rangle$ be two CSS such that $\langle \mathcal{F}, \mathcal{C} \rangle \preccurlyeq \langle \mathcal{F}', \mathcal{C}' \rangle$. Let $(\mathbb{S}\varphi : \mu, x) \in \sigma(\mathcal{F})$. We show that $(\mathbb{S}\varphi : \mu, x) \in \sigma(\mathcal{F}')$. There are $m \in L_r$ such that $x = \sigma(m)$ and $(\mathbb{S}\varphi : \mu, m) \in \mathcal{F}$. As

$\langle \mathcal{F}, \mathcal{C} \rangle \preccurlyeq \langle \mathcal{F}', \mathcal{C}' \rangle$ then $(\mathbb{S}\varphi : \mu, m) \in \mathcal{F}'$. Then $(\mathbb{S}\varphi : \mu, x) \in \sigma(\mathcal{F}')$. Hence $\sigma(\mathcal{F}) \subseteq \sigma(\mathcal{F}')$. The proof for $\sigma(\mathcal{C}) \subseteq \sigma(\mathcal{C}')$ is similar. We can conclude that $\langle \sigma(\mathcal{F}), \sigma(\mathcal{C}) \rangle \preccurlyeq \langle \sigma(\mathcal{F}'), \sigma(\mathcal{C}') \rangle$.

□

Proposition G.9. Let σ be a substitution, the following properties hold:

— $\sigma(\mathcal{F}) \cup \{(\mathbb{S}\varphi : \mu, \sigma(x))\} = \sigma(\mathcal{F} \cup \{(\mathbb{S}\varphi : \mu, x)\})$
— $\sigma(\mathcal{C}) \cup \{\sigma(x) \simeq \sigma(y)\} = \sigma(\mathcal{C} \cup \{x \simeq y\})$
— $\sigma(\mathcal{C}) \cup \{\sigma(x) \doteqdot^{\mu}_{u} \sigma(y)\} = \sigma(\mathcal{C} \cup \{x \doteqdot^{\mu}_{u} y\})$
— $\sigma(\mathcal{C}) \cup \{\triangleright\mu\} = \sigma(\mathcal{C} \cup \{\triangleright\mu\})$

Proof. This proof is left to the reader. □

Proposition G.10. Any $\preccurlyeq$-closed consistency property can be extended into a $\preccurlyeq$-closed alternate consistency property.

Proof. Let $\mathcal{P}$ be a $\preccurlyeq$-closed consistency property. Let $\mathcal{P}^+$ defined by:

$$\langle \mathcal{F}, \mathcal{C} \rangle \in \mathcal{P}^+ \text{ iff } \langle \sigma(\mathcal{F}), \sigma(\mathcal{C}) \rangle \in \mathcal{P}$$

for a substitution $\sigma : \gamma_r \cup \{\varepsilon\} \longrightarrow \gamma_r \cup \{\varepsilon\}$.
We remark that $\mathcal{P} \subseteq \mathcal{P}^+$ (by considering the identity substitution).
We show that $\mathcal{P}^+$ is $\preccurlyeq$-closed. Let $\langle \mathcal{F}, \mathcal{C} \rangle \in \mathcal{P}^+$ and $\langle \mathcal{F}', \mathcal{C}' \rangle \preccurlyeq \langle \mathcal{F}, \mathcal{C} \rangle$. As $\langle \mathcal{F}, \mathcal{C} \rangle \in \mathcal{P}^+$, there is a substitution σ such that $\langle \sigma(\mathcal{F}), \sigma(\mathcal{C}) \rangle \in \mathcal{P}$. By Proposition G.8, $\langle \sigma(\mathcal{F}'), \sigma(\mathcal{C}') \rangle \preccurlyeq \langle \sigma(\mathcal{F}), \sigma(\mathcal{C}) \rangle$. As $\mathcal{P}$ is $\preccurlyeq$-closed then $\langle \sigma(\mathcal{F}'), \sigma(\mathcal{C}') \rangle \in \mathcal{P}$. Thus $\langle \mathcal{F}', \mathcal{C}' \rangle \in \mathcal{P}^+$.
We show now that $\mathcal{P}^+$ is an alternate consistency property. Let $\langle \mathcal{F}, \mathcal{C} \rangle \in \mathcal{P}^+$. By definition, there exists a substitution σ such that $\langle \sigma(\mathcal{F}), \sigma(\mathcal{C}) \rangle \in \mathcal{P}$.

— We suppose that $(\mathbb{T}p : \mu, x) \in \mathcal{F}$ and $(\mathbb{F}p : \kappa, y) \in \mathcal{F}$ and $x \simeq y \in \overline{\mathcal{C}}$. By definition and Corollary G.7, $(\mathbb{T}p : \mu, \sigma(x)) \in \sigma(\mathcal{F})$ and $(\mathbb{F}p : \kappa, \sigma(y)) \in \sigma(\mathcal{F})$ and $\sigma(x) \simeq \sigma(y) \in \overline{\sigma(\mathcal{C})}$. It is contradictory because $\mathcal{P}$ is a consistency property.
— We suppose that $(\mathbb{T}K_u\varphi : \mu, x) \in \mathcal{F}$. Then $(\mathbb{T}K_u\varphi : \mu, \sigma(x)) \in \sigma(\mathcal{F})$. Let $y \in L_r$ such that $x \doteqdot^{\mu}_{u} y \in \overline{\mathcal{C}}$. By Corollary G.7, $\sigma(x \doteqdot^{\mu}_{u} y) \in \overline{\sigma(\mathcal{C})}$. By definition of substitution, $\sigma(x) \doteqdot^{\mu}_{u} \sigma(y) \in \overline{\sigma(\mathcal{C})}$. Moreover as $\mathcal{P}$ is a consistency property then $\langle \sigma(\mathcal{F}) \cup \{(\mathbb{T}\varphi : \mu, \sigma(y))\}, \sigma(\mathcal{C}) \rangle \in \mathcal{P}$. By Proposition G.9, $\langle \sigma(\mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, y)\}), \sigma(\mathcal{C}) \rangle \in \mathcal{P}$. Hence $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, y)\}, \mathcal{C} \rangle \in \mathcal{P}^+$.
— We suppose that $(\mathbb{F}K_u\varphi : \mu, x) \in \mathcal{F}$. Then $(\mathbb{F}K_u\varphi : \mu, \sigma(x)) \in \sigma(\mathcal{F})$. Let $c_i \in \gamma_r \setminus \mathcal{A}_r(\mathcal{C})$. As $\mathcal{P}$ is a consistency property then there exists $c' \in \gamma_r \setminus \mathcal{A}_r(\sigma(\mathcal{C}))$ such that $\langle \sigma(\mathcal{F}) \cup \{(\mathbb{F}\varphi : \mu, c')\}, \sigma(\mathcal{C}) \cup \{x \doteqdot^{\mu}_{u} c'\} \rangle \in \mathcal{P}$. Let $\sigma' = \sigma[c_i \mapsto c']$. As $c_i \notin \mathcal{A}_r(\mathcal{C})$ then $\sigma(\mathcal{C}) = \sigma'(\mathcal{C})$. Moreover, by Proposition 4.10, $c_i \notin \mathcal{A}_r(\overline{\mathcal{C}})$. Then, by property (P_{css}), c_i does not occur in $\mathcal{F}$. Thus $\sigma(\mathcal{F}) = \sigma'(\mathcal{F})$ and $\sigma(x) = \sigma'(x)$. Then $\langle \sigma'(\mathcal{F}) \cup \{(\mathbb{F}\varphi : \mu, \sigma'(c_i))\}, \sigma'(\mathcal{C}) \cup \{\sigma'(x) \doteqdot^{\mu}_{u} \sigma'(c_i)\} \rangle \in \mathcal{P}$. By Proposition G.9, $\langle \sigma'(\mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, c_i)\}), \sigma'(\mathcal{C} \cup \{x \doteqdot^{\mu}_{u} c_i\}) \rangle \in \mathcal{P}$. Hence $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, c_i)\}, \mathcal{C} \cup \{x \doteqdot^{\mu}_{u} c_i\} \rangle \in \mathcal{P}^+$.
— We suppose that $(\mathbb{T}[\varphi]\psi : \mu, x) \in \mathcal{F}$. Then, $(\mathbb{T}[\varphi]\psi : \mu, \sigma(x)) \in \sigma(\mathcal{F})$. As $\mathcal{P}$ is a consistency property then $\langle \sigma(\mathcal{F}) \cup \{(\mathbb{F}\varphi : \mu, \sigma(x))\}, \sigma(\mathcal{C}) \rangle \in \mathcal{P}$ or $\langle \sigma(\mathcal{F}) \cup \{(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!], \sigma(x))\}, \sigma(\mathcal{C}) \cup \{\triangleright\mu \oplus [\![\varphi]\!]\} \rangle \in \mathcal{P}$. By Proposition G.9, $\langle \sigma(\mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, x)\}), \sigma(\mathcal{C}) \rangle \in \mathcal{P}$ or $\langle \sigma(\mathcal{F} \cup \{(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!], x)\}), \sigma(\mathcal{C} \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}) \rangle \in \mathcal{P}$. Hence $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, x)\}, \mathcal{C} \rangle \in \mathcal{P}^+$ or $\langle \mathcal{F} \cup \{(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!], x)\}, \mathcal{C} \cup \{\triangleright\mu \oplus [\![\varphi]\!]\} \rangle \in \mathcal{P}^+$.

— We suppose that $(\mathbb{F}[\varphi]\psi : \mu, x) \in \mathcal{F}$. By definition, $(\mathbb{F}[\varphi]\psi : \mu, \sigma(x)) \in \sigma(\mathcal{F})$. As $\mathcal{P}$ is a consistency property then $\langle \sigma(\mathcal{F}) \cup \{(\mathbb{T}\varphi : \mu, \sigma(x)), (\mathbb{F}\psi : \mu \oplus [\![\varphi]\!], \sigma(x))\}, \sigma(\mathcal{C}) \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle \in \mathcal{P}$. By Proposition G.9, $\langle \sigma(\mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, x), (\mathbb{F}\psi : \mu \oplus [\![\varphi]\!], x)\}), \sigma(\mathcal{C} \cup \{\triangleright\mu \oplus [\![\varphi]\!]\})\rangle \in \mathcal{P}$. Hence $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, x), (\mathbb{F}\psi : \mu \oplus [\![\varphi]\!], x)\}, \mathcal{C} \cup \{\triangleright\mu \oplus [\![\varphi]\!]\}\rangle \in \mathcal{P}^+$.

— Other cases are proved similarly.

□

Proposition G.11. Any $\preccurlyeq$-closed alternate consistency property can be extended into a $\preccurlyeq$-closed alternate consistency property of finite character.

Proof. Let $\mathcal{P}$ be a $\preccurlyeq$-closed alternate consistency property. Let $\mathcal{P}^{fc}$ defined by:

$$\langle \mathcal{F}, \mathcal{C}\rangle \in \mathcal{P}^{fc} \text{ iff } \langle \mathcal{F}_f, \mathcal{C}_f\rangle \in \mathcal{P} \text{ for all } \langle \mathcal{F}_f, \mathcal{C}_f\rangle \preccurlyeq_f \langle \mathcal{F}, \mathcal{C}\rangle$$

We show that $\mathcal{P} \subseteq \mathcal{P}^{fc}$. Let $\langle \mathcal{F}, \mathcal{C}\rangle \in \mathcal{P}$. As $\mathcal{P}$ is $\preccurlyeq$-closed then $\mathcal{P}$ is $\preccurlyeq_f$-closed. Thus, for any $\langle \mathcal{F}_f, \mathcal{C}_f\rangle \preccurlyeq_f \langle \mathcal{F}, \mathcal{C}\rangle$, we have $\langle \mathcal{F}_f, \mathcal{C}_f\rangle \in \mathcal{P}$. Then $\langle \mathcal{F}, \mathcal{C}\rangle \in \mathcal{P}^{fc}$ by definition.
We show that $\mathcal{P}^{fc}$ is $\preccurlyeq$-closed. Let $\langle \mathcal{F}, \mathcal{C}\rangle \in \mathcal{P}^{fc}$. Let $\langle \mathcal{F}', \mathcal{C}'\rangle \preccurlyeq \langle \mathcal{F}, \mathcal{C}\rangle$. We show that $\langle \mathcal{F}', \mathcal{C}'\rangle \in \mathcal{P}^{fc}$. Let $\langle \mathcal{F}'_f, \mathcal{C}'{}_f\rangle \preccurlyeq_f \langle \mathcal{F}', \mathcal{C}'\rangle$. Then $\langle \mathcal{F}'_f, \mathcal{C}'_f\rangle \preccurlyeq_f \langle \mathcal{F}, \mathcal{C}\rangle$. Thus, we have $\langle \mathcal{F}'_f, \mathcal{C}'_f\rangle \in \mathcal{P}$, by definition. Hence $\langle \mathcal{F}', \mathcal{C}'\rangle \in \mathcal{P}^{fc}$ by definition.
We show that $\mathcal{P}^{fc}$ is of finite character. Let $\langle \mathcal{F}, \mathcal{C}\rangle$ be a CSS. We suppose that for all $\langle \mathcal{F}_f, \mathcal{C}_f\rangle \preccurlyeq_f \langle \mathcal{F}, \mathcal{C}\rangle$, $\langle \mathcal{F}_f, \mathcal{C}_f\rangle \in \mathcal{P}^{fc}$ holds. We show that $\langle \mathcal{F}, \mathcal{C}\rangle \in \mathcal{P}^{fc}$. By definition, for all $\langle \mathcal{F}'_f, \mathcal{C}'_f\rangle \preccurlyeq_f \langle \mathcal{F}_f, \mathcal{C}_f\rangle$ we have $\langle \mathcal{F}'_f, \mathcal{C}'_f\rangle \in \mathcal{P}$. In particular, as $\langle \mathcal{F}_f, \mathcal{C}_f\rangle \preccurlyeq_f \langle \mathcal{F}_f, \mathcal{C}_f\rangle$ then $\langle \mathcal{F}_f, \mathcal{C}_f\rangle \in \mathcal{P}$. Therefore, we have $\langle \mathcal{F}, \mathcal{C}\rangle \in \mathcal{P}^{fc}$.
We now show that $\mathcal{P}^{fc}$ is an alternate consistency property. Let $\langle \mathcal{F}, \mathcal{C}\rangle \in \mathcal{P}^{fc}$.

— We suppose that $(\mathbb{T}p : \mu, x) \in \mathcal{F}$ and $(\mathbb{F}p : \kappa, y) \in \mathcal{F}$ and $x \simeq y \in \overline{\mathcal{C}}$. By compactness (Lemma 4.11), there exists $\mathcal{C}_f \subseteq \mathcal{C}$ such that $\mathcal{C}_f$ is finite and $x \simeq y \in \overline{\mathcal{C}_f}$. Then $\langle \{(\mathbb{T}p : \mu, x), (\mathbb{F}p : \kappa, y)\}, \mathcal{C}_f\rangle$ is a CSS (satisfies the property (P_{css})) and we remark that $\langle \{(\mathbb{T}p : \mu, x), (\mathbb{F}p : \kappa, y)\}, \mathcal{C}_f\rangle \preccurlyeq_f \langle \mathcal{F}, \mathcal{C}\rangle$. Thus, by definition $\langle \{(\mathbb{T}p : \mu, x), (\mathbb{F}p : \kappa, y)\}, \mathcal{C}_f\rangle \in \mathcal{P}$. But this is contradictory because $\mathcal{P}$ is an alternate consistency property.

— We suppose that $(\mathbb{T}K_u\varphi : \mu, x) \in \mathcal{F}$. Let $y \in L_r$ such that $x \eqcirc^{\mu}_{u} y \in \overline{\mathcal{C}}$. We suppose that $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, y)\}, \mathcal{C}\rangle \notin \mathcal{P}^{fc}$. Thus, by definition of $\mathcal{P}^{fc}$, there exists $\langle \mathcal{F}_f, \mathcal{C}_f\rangle \preccurlyeq_f \langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, y)\}, \mathcal{C}\rangle$ such that $\langle \mathcal{F}_f, \mathcal{C}_f\rangle \notin \mathcal{P}$. By compactness there exists a finite $\mathcal{C}_1 \subseteq \mathcal{C}$ such that $x \eqcirc^{\mu}_{u} y \in \overline{\mathcal{C}_1}$. Let $\mathcal{F}'_f = \mathcal{F}_f \setminus \{(\mathbb{T}\varphi : \mu, y)\} \cup \{(\mathbb{T}K_u\varphi : \mu, x)\}$. Let $\mathcal{C}'_f = \mathcal{C}_f \cup \mathcal{C}_1$. Then $\langle \mathcal{F}'_f, \mathcal{C}'_f\rangle$ is a finite CSS and $\langle \mathcal{F}'_f, \mathcal{C}'_f\rangle \preccurlyeq_f \langle \mathcal{F}, \mathcal{C}\rangle$. Thus, by definition of $\mathcal{P}^{fc}$, $\langle \mathcal{F}'_f, \mathcal{C}'_f\rangle \in \mathcal{P}$. As $(\mathbb{T}K_u\varphi : \mu, x) \in \mathcal{F}'_f$, $x \eqcirc^{\mu}_{u} y \in \overline{\mathcal{C}'_f}$ and as $\mathcal{P}$ is an alternate consistency property then $\langle \mathcal{F}'_f \cup \{(\mathbb{T}\varphi : \mu, y)\}, \mathcal{C}'_f\rangle \in \mathcal{P}$. We can remark that $\langle \mathcal{F}_f, \mathcal{C}_f\rangle \preccurlyeq \langle \mathcal{F}'_f \cup \{(\mathbb{T}\varphi : \mu, y)\}, \mathcal{C}'_f\rangle$ holds. As $\mathcal{P}$ is $\preccurlyeq$-closed then $\langle \mathcal{F}_f, \mathcal{C}_f\rangle \in \mathcal{P}$. But, this is contradictory. Therefore $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, y)\}, \mathcal{C}\rangle \in \mathcal{P}^{fc}$.

— We suppose that $(\mathbb{F}K_u\varphi : \mu, x) \in \mathcal{F}$. Let $c_i \in \gamma_r \setminus \mathcal{A}_r(\mathcal{C})$. We show that $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, c_i)\}, \mathcal{C} \cup \{x \eqcirc^{\mu}_{u} c_i\}\rangle \in \mathcal{P}^{fc}$. Let $\langle \mathcal{F}_f, \mathcal{C}_f\rangle \preccurlyeq_f \langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, c_i)\}, \mathcal{C} \cup \{x \eqcirc^{\mu}_{u} c_i\}\rangle$. Let $\mathcal{F}'_f = \mathcal{F}_f \setminus \{(\mathbb{F}\varphi : \mu, c_i)\} \cup \{(\mathbb{F}K_u\varphi : \mu, x)\}$. As $\langle \mathcal{F}, \mathcal{C}\rangle$ is a CSS and $\mathcal{F}'_f \subseteq \mathcal{F}$ then $\langle \mathcal{F}'_f, \mathcal{C}\rangle$ is a CSS. By Proposition 4.13, there exists $\mathcal{C}_1 \subseteq \mathcal{C}$ such that $\mathcal{C}_1$ is finite and $\langle \mathcal{F}'_f, \mathcal{C}_1\rangle$ is a CSS. Let $\mathcal{C}'_f = \mathcal{C}_f \setminus \{x \eqcirc^{\mu}_{u} c_i\} \cup \mathcal{C}_1$. Then $\langle \mathcal{F}'_f, \mathcal{C}'_f\rangle$ is a finite CSS and $\langle \mathcal{F}'_f, \mathcal{C}'_f\rangle \preccurlyeq_f \langle \mathcal{F}, \mathcal{C}\rangle$. Thus, by definition, $\langle \mathcal{F}'_f, \mathcal{C}'_f\rangle \in \mathcal{P}$. Also, as $\mathcal{C}'_f \subseteq \mathcal{C}$ then $\mathcal{A}_r(\mathcal{C}'_f) \subseteq \mathcal{A}_r(\mathcal{C})$. As $\mathcal{P}$ is an alternate consistency property, as $(\mathbb{F}K_u\varphi : \mu, x) \in \mathcal{F}'_f$ and as $c_i \in \gamma_r \setminus \mathcal{A}_r(\mathcal{C}'_f)$ then $\langle \mathcal{F}'_f \cup \{(\mathbb{F}\varphi : \mu, c_i)\}, \mathcal{C}'_f \cup \{x \eqcirc^{\mu}_{u} c_i\}\rangle \in \mathcal{P}$. As $\mathcal{P}$ is $\preccurlyeq$-closed and as $\langle \mathcal{F}_f, \mathcal{C}_f\rangle \preccurlyeq \langle \mathcal{F}'_f \cup \{(\mathbb{F}\varphi : c_i, \}), \mathcal{C}'_f \cup \{x \eqcirc^{\mu}_{u} c_i\}\rangle$ then

$\langle \mathcal{F}_f, \mathcal{C}_f \rangle \in \mathcal{P}$. This being true for any $\langle \mathcal{F}_f, \mathcal{C}_f \rangle \preccurlyeq_f \langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, c_i)\}, \mathcal{C} \cup \{x =^{\mu}_{u} c_i\} \rangle$, then $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, c_i)\}, \mathcal{C} \cup \{x =^{\mu}_{u} c_i\} \rangle \in \mathcal{P}^{fc}$.

— We suppose that $(\mathbb{T}[\varphi]\psi : \mu, x) \in \mathcal{F}$. We suppose that $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, x)\}, \mathcal{C} \rangle \notin \mathcal{P}^{fc}$ and $\langle \mathcal{F} \cup \{(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!], x)\}, \mathcal{C} \cup \{\triangleright \mu \oplus [\![\varphi]\!]\} \rangle \notin \mathcal{P}^{fc}$. By definition of $\mathcal{P}^{fc}$, there exist $\langle \mathcal{F}^A_f, \mathcal{C}^A_f \rangle \preccurlyeq_f \langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, x)\}, \mathcal{C} \rangle$ and $\langle \mathcal{F}^B_f, \mathcal{C}^B_f \rangle \preccurlyeq_f \langle \mathcal{F} \cup \{(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!], x)\}, \mathcal{C} \cup \{\triangleright \mu \oplus [\![\varphi]\!]\} \rangle$ such that $\langle \mathcal{F}^A_f, \mathcal{C}^A_f \rangle \notin \mathcal{P}$ and $\langle \mathcal{F}^B_f, \mathcal{C}^B_f \rangle \notin \mathcal{P}$. Let $\mathcal{F}'_f = \mathcal{F}^A_f \setminus \{(\mathbb{F}\varphi : \mu, x)\} \cup \mathcal{F}^B_f \setminus \{(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!], x)\} \cup \{(\mathbb{T}[\varphi]\psi : \mu, x)\}$. As $\langle \mathcal{F}, \mathcal{C} \rangle$ is a CSS and $\mathcal{F}'_f \subseteq \mathcal{F}$ then $\langle \mathcal{F}'_f, \mathcal{C} \rangle$ is a CSS. By Proposition 4.13, there exists $\mathcal{C}_1 \subseteq \mathcal{C}$ such that $\mathcal{C}_1$ is finite and $\langle \mathcal{F}'_f, \mathcal{C}_1 \rangle$ is a CSS. Let $\mathcal{C}'_f = \mathcal{C}^A_f \cup \mathcal{C}^B_f \setminus \{\triangleright \mu \oplus [\![\varphi]\!]\} \cup \mathcal{C}_1$. Then $\langle \mathcal{F}'_f, \mathcal{C}'_f \rangle$ is a finite CSS and $\langle \mathcal{F}'_f, \mathcal{C}'_f \rangle \preccurlyeq_f \langle \mathcal{F}, \mathcal{C} \rangle$. Thus, by definition of $\mathcal{P}^{fc}$, $\langle \mathcal{F}'_f, \mathcal{C}'_f \rangle \in \mathcal{P}$. As $(\mathbb{T}[\varphi]\psi : \mu, x) \in \mathcal{F}'_f$ and as $\mathcal{P}$ is an alternate consistency property then $\langle \mathcal{F}'_f \cup \{(\mathbb{F}\varphi : \mu, x)\}, \mathcal{C}'_f \rangle \in \mathcal{P}$ or $\langle \mathcal{F}'_f \cup \{(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!], x)\}, \mathcal{C}'_f \cup \{\triangleright \mu \oplus [\![\varphi]\!]\} \rangle \in \mathcal{P}$. We can remark that $\langle \mathcal{F}^A_f, \mathcal{C}^A_f \rangle \preccurlyeq \langle \mathcal{F}'_f \cup \{(\mathbb{F}\varphi : \mu, x)\}, \mathcal{C}'_f \rangle$ and $\langle \mathcal{F}^B_f, \mathcal{C}^B_f \rangle \preccurlyeq \langle \mathcal{F}'_f \cup \{(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!], x)\}, \mathcal{C}'_f \cup \{\triangleright \mu \oplus [\![\varphi]\!]\} \rangle$ hold. As $\mathcal{P}$ is $\preccurlyeq$-closed then $\langle \mathcal{F}^A_f, \mathcal{C}^A_f \rangle \in \mathcal{P}$ and $\langle \mathcal{F}^B_f, \mathcal{C}^B_f \rangle \in \mathcal{P}$. But, this is contradictory. Therefore $\langle \mathcal{F} \cup \{(\mathbb{F}\varphi : \mu, x)\}, \mathcal{C} \rangle \in \mathcal{P}^{fc}$ or $\langle \mathcal{F} \cup \{(\mathbb{T}\psi : \mu \oplus [\![\varphi]\!], x)\}, \mathcal{C} \cup \{\triangleright \mu \oplus [\![\varphi]\!]\} \rangle \in \mathcal{P}^{fc}$.

— We suppose that $(\mathbb{F}[\varphi]\psi : \mu, x) \in \mathcal{F}$. We suppose that $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, x), (\mathbb{F}\psi : \mu \oplus [\![\varphi]\!], x)\}, \mathcal{C} \cup \{\triangleright \mu \oplus [\![\varphi]\!]\} \rangle \notin \mathcal{P}^{fc}$. By definition of $\mathcal{P}^{fc}$, there exists $\langle \mathcal{F}_f, \mathcal{C}_f \rangle \preccurlyeq_f \langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, x), (\mathbb{F}\psi : \mu \oplus [\![\varphi]\!], x)\}, \mathcal{C} \cup \{\triangleright \mu \oplus [\![\varphi]\!]\} \rangle$ such that $\langle \mathcal{F}_f, \mathcal{C}_f \rangle \notin \mathcal{P}$. Let $\mathcal{F}'_f = \mathcal{F}_f \setminus \{(\mathbb{T}\varphi : \mu, x), (\mathbb{F}\psi : \mu \oplus [\![\varphi]\!], x)\} \cup \{(\mathbb{F}[\varphi]\psi : \mu, x)\}$. As $\langle \mathcal{F}, \mathcal{C} \rangle$ is a CSS and $\mathcal{F}'_f \subseteq \mathcal{F}$ then $\langle \mathcal{F}'_f, \mathcal{C} \rangle$ is a CSS. By Proposition 4.13, there exists $\mathcal{C}_1 \subseteq \mathcal{C}$ such that $\mathcal{C}_1$ is finite and $\langle \mathcal{F}'_f, \mathcal{C}_1 \rangle$ is a CSS. Let $\mathcal{C}'_f = \mathcal{C}_f \setminus \{\triangleright \mu \oplus [\![\varphi]\!]\} \cup \mathcal{C}_1$. Then $\langle \mathcal{F}'_f, \mathcal{C}'_f \rangle$ is a finite CSS and $\langle \mathcal{F}'_f, \mathcal{C}'_f \rangle \preccurlyeq_f \langle \mathcal{F}, \mathcal{C} \rangle$. Thus, by definition of $\mathcal{P}^{fc}$, $\langle \mathcal{F}'_f, \mathcal{C}'_f \rangle \in \mathcal{P}$. As $(\mathbb{F}[\varphi]\psi : \mu, x) \in \mathcal{F}'_f$ and as $\mathcal{P}$ is an alternate consistency property then $\langle \mathcal{F}'_f \cup \{(\mathbb{T}\varphi : \mu, x), (\mathbb{F}\psi : \mu \oplus [\![\varphi]\!], x)\}, \mathcal{C}'_f \cup \{\triangleright \mu \oplus [\![\varphi]\!]\} \rangle \in \mathcal{P}$. We can remark that $\langle \mathcal{F}_f, \mathcal{C}_f \rangle \preccurlyeq \langle \mathcal{F}'_f \cup \{(\mathbb{T}\varphi : \mu, x), (\mathbb{F}\psi : \mu \oplus [\![\varphi]\!], x)\}, \mathcal{C}'_f \cup \{\triangleright \mu \oplus [\![\varphi]\!]\} \rangle$ holds. As $\mathcal{P}$ is $\preccurlyeq$-closed then $\langle \mathcal{F}_f, \mathcal{C}_f \rangle \in \mathcal{P}$. But, this is contradictory. Therefore $\langle \mathcal{F} \cup \{(\mathbb{T}\varphi : \mu, x), (\mathbb{F}\psi : \mu \oplus [\![\varphi]\!], x)\}, \mathcal{C} \cup \{\triangleright \mu \oplus [\![\varphi]\!]\} \rangle \in \mathcal{P}^{fc}$.

— Other cases are proved similarly.

□

Lemma 4.30. There exists an oracle which contains every finite CSS for which there exists no closed tableau.

Proof. We consider the set of the finite CSS for which there is no closed tableau. By Proposition G.3, this set is a consistency property. By Proposition G.4, we can extend it into a $\preccurlyeq$-close consistency property. By Proposition G.10, we can extend it into a $\preccurlyeq$-close alternate consistency property. Finally, by Proposition G.11, we can extend it into a $\preccurlyeq$-close alternate consistency property of finite character. By conditions 6 to 28 of the alternate consistency property, this set is saturated. And by conditions 1 to 5 this is a set of non closed CSS. Then this set is an oracle. As this set is an extension of every finite CSS for which there is no closed tableau, we can conclude that there exists an oracle which contains every finite CSS for which there is no closed tableau.

□