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Intuitionistic non-normal modal logics: A general framework

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Abstract We define a family of intuitionistic non-normal modal logics; they can bee seen as intuitionistic counterparts of classical ones. We first consider monomodal logics, which contain only Necessity or Possibility. We then consider the more important case of bimodal logics, which contain both modal operators. In this case we define several interactions between Necessity and Possibility of increasing strength, although weaker than duality. We thereby obtain a lattice of 24 distinct bimodal logics. For all logics we provide both a Hilbert axiomatisation and a cut-free sequent calculus, on its basis we also prove their decidability. We then define a semantic characterisation of our logics in terms of neighbourhood models containing two distinct neighbourhood functions corresponding to the two modalities. Our semantic framework captures modularly not only our systems but also already known intuitionistic non-normal modal logics such as Constructive K (CK) and the propositional fragment of Wijesekera's Constructive Concurrent Dynamic Logic.

1 Introduction

Both intuitionistic modal logic and non-normal modal logic have been studied for a long time. The study of modalities with an intuitionistic basis goes back to Fitch in the late 40s [12] and has led to an important stream of research. We can very schematically identify two traditions: so-called *intuitionistic modal logics* versus *constructive modal logics*. Intuitionistic modal logics have been systematised by Simpson [38], whose main goal is to define an analogous of classical modalities justified from an intuitionistic point of view. On the other hand, constructive modal logics are mainly motivated by their applications to

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computer science, such as the type-theoretic interpretations (Curry–Howard correspondence, typed lambda calculi), verification and knowledge representation[1], but also by their mathematical semantics (Goldblatt [17]).

On the other hand, non-normal modal logics have been strongly motivated on a philosophical and epistemic ground. They are called "non-normal" as they do not satisfy all the axioms and rules of the minimal normal modal logic K. They have been studied since the seminal works of Scott, Montague, Lemmon, and Chellas ([37], [35], [6], see Pacuit [36] for a survey), and can be seen as generalisations of standard modal logics. They have found an interest in several areas such as epistemic and deontic reasoning, reasoning about games, and reasoning about probabilistic notions such as "truth in most of the cases" (for the latter interpretation see *e.g.* Askounis *et al.* [3]). While the semantics of these logics has been widely investigated in the seminal works mentioned above, recently proof systems for these logics have been proposed in Lavendhomme and Lucas [26], Gilbert and Maffezioli [16], Negri [28], Dalmonte *et al.* [8], and Lellmann and Pimentel [27].

Although the two areas have grown up seemingly without any interaction, it can be noticed that some intuitionistic or constructive modal logics investigated in the literature contain non-normal modalities. The prominent example is the Constructive Concurrent Dynamic Logic (CCDL) proposed by Wijesekera [42], whose propositional fragment (that we call $\mathsf{CCDL}^{\mathsf{p}}$) has been recently investigated by Kojima [24]. This logic has a normal $\Box$ modality and a non-normal $\Diamond$ modality, where $\Diamond$ does not distribute over the $\vee$, that is

$$(\mathsf{C}_\Diamond) \qquad \Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$$

is not valid. The original motivation by Wijesekera comes from Constructive Concurrent Dynamic Logic, but the logic has also an interesting epistemic interpretation in terms of internal/external observers proposed by Kojima. A related system is Constructive K (CK), that has been proposed by Bellin *et al.* [4] and further investigated by Mendler and de Paiva [33], Mendler and Scheele [34]. This system not only rejects $\mathsf{C}_\Diamond$, but also its nullary version $\Diamond\bot \supset \bot$ ($\mathsf{N}_\Diamond$). In contrast all these systems consider a normal interpretation of $\Box$ so that

$$\Box(A \wedge B) \supset\subset (\Box A \wedge \Box B)$$

is always assumed. A further example is Propositional Lax Logic (PLL) by Fairtlough and Mendler [9], an intuitionistic monomodal logic for hardware verification where the modality does not validate the rule of necessitation.

Finally, all intuitionistic modal logics reject the interdefinability of the two operators:

$$\Box A \supset\subset \neg\Diamond\neg A$$

and its boolean equivalents.

To the best of our knowledge, no systematic investigation of non-normal modalities with an intuitionistic base has been carried out so far. Our aim is

[1] For a recent survey see Stewart *et al.* [40] and references therein.

to lay down a general framework which can accommodate in a uniform way intuitionistic counterparts of the classical cube of non-normal modal logics, as well as $\mathsf{CCDL^p}$ and CK mentioned above. In the present investigation, we deliberately restrict our attention to systems which are close to *constructive modal logics*: all of them reject the axiom $\mathsf{C}_\Diamond$, so that the stronger system we shall consider is $\mathsf{CCDL^p}$.[2] As we shall see, the adoption of an intuitionistic base leads to a *finer* analysis of non-normal modalities than in the classical case. In addition to the motivations for classical non-normal modal logics briefly recalled above, an intuitionistic interpretation of non-normal modalities may be justified by more specific interpretations, of which we mention two.

The **deontic interpretation**: The standard interpretation of deontic operators $\Box$ (Obligatory), $\Diamond$ (Permitted) is normal: but it has been known for a long time that the normal interpretation is problematic when dealing for instance with "Contrary to duty obligations".[3] One solution is to adopt a non-normal interpretation, rejecting in particular the monotonicity principle (from $A \supset B$ is valid infer $\Box A \supset \Box B$). Moreover, a constructive reading of the deontic modalities would further reject their interdefinability: one may require that the permission of A must be justified explicitly or positively (say by a proof from a corpus of norms) and not just established by the fact that $\neg A$ is not obligatory. This distinction is somewhat close to the distinction by von Wright [47] between *weak* and *strong* permissions, also widely discussed by Hansson [18] who distinguish *explicit* and *implicit* permissions on one side (the strong ones) from *tacit* permissions on the other side (weak ones); recently in Anglberger *et al.* [1] it is proposed a deontic logic where the modalities are non-normal and not interdefinable.

The **contextual interpretation**: A contextual reading of the modal operators is proposed in Mendler and de Paiva [33]. In this interpretation $\Box A$ is read as "A holds in all contexts" and $\Diamond A$ as "A holds in some context". This interpretation invalidates $\mathsf{C}_\Diamond$, while retaining the distribution of $\Box$ over conjunction ($\mathsf{C}_\Box$). But this contextual interpretation is not the only possible one. We can interpret $\Box A$ as A is "justified" (proved) in some context c, no matter what is meant by a context (for instance a knowledge base), and $\Diamond A$ as A is "compatible" (consistent) with every context. With this interpretation both operators would be non-normal as they would satisfy neither $\mathsf{C}_\Box$, nor $\mathsf{C}_\Diamond$.

As we said, our aim is to provide a general framework for non-normal modal logics with an intuitionistic base. However, in order to identify and restrain the family of logics of interest, we adopt some criteria, which partially coincide with Simpson's requirements (Simpson [38]):

- The modal logics should be conservative extensions of IPL.
- The disjunction property must hold.

[2] The extension of our framework to logics stronger than $\mathsf{CCDL^p}$, allowing $\mathsf{C}_\Diamond$ such as Simpson intuitionistic modal logic, is beyond the scope of this work; however in Section 8 we make some preliminary considerations in the perspective of a future extension of our results in this direction.

[3] For a survey on puzzles related to a normal interpretation of the deontic modalities see McNamara [32].

– The two modalities should not be interdefinable.
– We do not consider systems containing the controversial $\mathsf{C}_\Diamond$.

Our starting point is the study of *monomodal* systems, which extend IPL with either $\Box$ or $\Diamond$, but not both. We consider the monomodal logics corresponding to the classical cube generated by the weakest logic E extended with conditions M, N, C (with the exception of $\mathsf{C}_\Diamond$). We give an axiomatic characterisation of these logics and equivalent cut-free sequent systems similar to the one by Lavendhomme and Lucas [26] for the classical case.

Our main interest is however in logics which contain *both* $\Box$ and $\Diamond$, and allow some form of interaction between the two. Their interaction is always weaker than interdefinability. In order to define logical systems we take a proof-theoretical perspective: the existence of a simple cut-free system, as in the monomodal case, is our criterion to identify meaningful systems. *A system is considered if the combination of its sequent rules provides a cut-free system.*

It turns out that one can distinguish *three* degrees of interaction between $\Box$ and $\Diamond$, that are determined by answering the following question, for any two formulas A and B:

under which conditions are $\Box A$ *and* $\Diamond B$ *jointly inconsistent?*

Since there are *three* degrees of interaction, even the weakest classical logic E has *three* intuitionistic counterparts of increasing strength. When combined with M, N, C properties of the classical cube, we end up with a family of 24 distinct systems, instead of just 8 systems as in the classical case. All systems enjoy a cut-free calculus and, as we prove, an equivalent Hilbert axiomatisation. This shows that intuitionistic non-normal modal logic allows for finer distinctions, whence a richer theory than in the classical case.

The existence of a cut-free calculus for each of the logics has some important consequences: we can prove that all systems are indeed distinct, that all of them are "good" extensions of intuitionistic logic, and more importantly, all of them are decidable.

We then tackle the problem of giving a semantic characterisation of this family of logics. The natural setting is to consider an intuitionistic version of neighbourhood models for classical logics. Since we want to deal with the language containing both $\Box$ and $\Diamond$, we consider neighbourhood models containing *two* distinct neighbourhood functions $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$. As in standard intuitionistic models, they also contain a partial order on worlds. Different forms of interaction between the two modal operators correspond to different (but natural) conditions relating the two neighbourhood functions. By considering further closure conditions of neighbourhoods, analogous to the classical case, we can show that this semantics characterises *modularly* the full family of logics. Moreover we prove, through a filtration argument, that all logics have the *finite model property*, thereby obtaining a semantic proof of their decidability.

It is worth noticing that in the (easier) case of intuitionistic monomodal logic with only $\Box$ a similar semantics and a matching completeness theorem have been given by Goldblatt [17]. More recently, Goldblatt's semantics for the intuitionistic monomodal version of system E has been reformulated and extended to axiom T by Witczak [45].

But our neighbourhood models have a wider application than the characterisation of the family of logics mentioned above. We show that adding suitable *interaction conditions* between $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ we can capture $\mathsf{CCDL^p}$ as well as CK. We show this fact first directly by proving that both $\mathsf{CCDL^p}$ and CK are sound and complete with respect to our models satisfying an additional condition. We then prove the same result by relying on some pre-existing semantics of these two logics and by a mutual transformation of models. In case of $\mathsf{CCDL^p}$, there exists already a characterisation of it in terms of neighbourhood models, given by Kojima [24], although the type of models is different. In particular, Kojima's models contain only one neighbourhood function.

The case of CK is more complicated, whence more interesting: this logic is characterised by a relational semantics defined in terms of Kripke models of a *peculiar* nature: they contain "fallible" worlds, *i.e.* worlds which force $\bot$. We are able to show directly that relational models can be transformed into our neighbourhood models satisfying a specific interaction condition and *vice versa*.

All in all, we get that the well-known logic CK can be characterised by neighbourhood models, which are quite standard structures, alternative to non-standard Kripke models with fallible worlds. This fact provides further evidence in favour of our neighbourhood semantics as a versatile tool to analyse intuitionistic non-normal modal logics.

2 Classical non-normal modal logics

2.1 Hilbert systems

Classical non-normal modal logics are defined on a propositional modal language $\mathcal{L}$ based on a set *Atm* of countably many propositional variables. Formulas are given by the following grammar, where p ranges over *Atm*:

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid \Box A \mid \Diamond A.$$

We use A, B, C as metavariables for formulas of $\mathcal{L}$. $\top$, $\neg A$ and $A \supset\subset B$ are abbreviations for, respectively, $\bot \supset \bot$, $A \supset \bot$ and $(A \supset B) \wedge (B \supset A)$. We take both modal operators $\Box$ and $\Diamond$ as primitive (as well as all other connectives), as it will be convenient for the intuitionistic case. Their duality in classical modal logics is recovered by adding to any system one of the duality axioms $\mathsf{Dual}_\Box$ or $\mathsf{Dual}_\Diamond$ (Figure 1), which are equivalent in the classical setting.

The weakest classical non-normal modal logic E is defined in language $\mathcal{L}$ by extending classical propositional logic (CPL) with a duality axiom and rule $\mathsf{E}_\Box$, and it can be extended further by adding any combination of axioms $\mathsf{M}_\Box$, $\mathsf{C}_\Box$ and $\mathsf{N}_\Box$. We obtain in this way eight distinct systems (Figure 2), which compose the family of classical non-normal modal logics.

Equivalent axiomatisations for these systems are given by considering the modal axioms in the right-hand column of Figure 1(a). Thus, logic E could be

***a.* Modal axioms and rules defining non-normal modal logics**

$\mathsf{E}_\Box$	$\dfrac{A \supset B \qquad B \supset A}{\Box A \supset \Box B}$	$\mathsf{E}_\Diamond$	$\dfrac{A \supset B \qquad B \supset A}{\Diamond A \supset \Diamond B}$
$\mathsf{M}_\Box$	$\Box(A \wedge B) \supset \Box A$	$\mathsf{M}_\Diamond$	$\Diamond A \supset \Diamond(A \vee B)$
$\mathsf{C}_\Box$	$\Box A \wedge \Box B \supset \Box(A \wedge B)$	$\mathsf{C}_\Diamond$	$\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$
$\mathsf{N}_\Box$	$\Box \top$	$\mathsf{N}_\Diamond$	$\neg \Diamond \bot$

***b.* Duality axioms**

$\mathsf{Dual}_\Box$	$\Diamond A \supset\subset \neg\Box\neg A$	$\mathsf{Dual}_\Diamond$	$\Box A \supset\subset \neg\Diamond\neg A$

***c.* Further relevant modal axioms and rules**

$\mathsf{K}_\Box$	$\Box(A \supset B) \supset (\Box A \supset \Box B)$			$\mathsf{K}_\Diamond$	$\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$
Nec	$\dfrac{A}{\Box A}$	$\mathsf{Mon}_\Box$	$\dfrac{A \supset B}{\Box A \supset \Box B}$	$\mathsf{Mon}_\Diamond$	$\dfrac{A \supset B}{\Diamond A \supset \Diamond B}$

Fig. 1: Modal axioms.

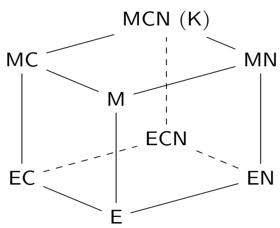

Fig. 2: The classical cube.

defined by extending CPL with axiom $\mathsf{Dual}_\Box$ and rule $\mathsf{E}_\Diamond$, and its extensions are given by adding combinations of axioms $\mathsf{M}_\Diamond$, $\mathsf{C}_\Diamond$ and $\mathsf{N}_\Diamond$.

It is worth recalling that axioms $\mathsf{M}_\Box$, $\mathsf{M}_\Diamond$ and $\mathsf{N}_\Box$ are syntactically equivalent with the rules $\mathsf{Mon}_\Box$, $\mathsf{Mon}_\Diamond$ and Nec, respectively, and that axiom $\mathsf{K}_\Box$ is derivable from $\mathsf{M}_\Box$ and $\mathsf{C}_\Box$. As a consequence, we have that the top system MCN is equivalent to the weakest classical normal modal logic K.

2.2 Neighbourhood semantics

The standard semantics for classical non-normal modal logics is based on the so-called neighbourhood (or minimal, or Scott-Montague) models.

Definition 1 A *neighbourhood model* is a triple $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set, $\mathcal{N}$ is a neighbourhood function $\mathcal{W} \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}))$, and $\mathcal{V}$ is a valuation function $\mathcal{W} \longrightarrow \mathcal{P}(Atm)$. A neighbourhood model is supplemented, closed under intersection, or contains the unit, if $\mathcal{N}$ satisfies the following properties:

If $\alpha \in \mathcal{N}(w)$ and $\alpha \subseteq \beta$, then $\beta \in \mathcal{N}(w)$	(Supplementation);
If $\alpha, \beta \in \mathcal{N}(w)$, then $\alpha \cap \beta \in \mathcal{N}(w)$	(Closure under intersection);
$\mathcal{W} \in \mathcal{N}(w)$ for all $w \in \mathcal{W}$	(Containing the unit).

The forcing relation $w \Vdash A$ is defined inductively as follows:

$w \Vdash p$	iff	$p \in \mathcal{V}(w)$;
$w \nVdash \bot$;		
$w \Vdash B \wedge C$	iff	$w \Vdash A$ and $w \Vdash B$;
$w \Vdash B \vee C$	iff	$w \Vdash A$ or $w \Vdash B$;
$w \Vdash B \supset C$	iff	$w \Vdash B$ implies $w \Vdash C$;
$w \Vdash \Box B$	iff	$[B] \in \mathcal{N}(w)$;
$w \Vdash \Diamond B$	iff	$\mathcal{W} \setminus [B] \notin \mathcal{N}(w)$;

where $[B]$ denotes the set $\{v \in \mathcal{W} \mid v \Vdash B\}$, called the *truth set* of B.

We can also recall that in the supplemented case, the forcing conditions for modal formulas are equivalent to the following ones:

$w \Vdash \Box B$	iff	there is $\alpha \in \mathcal{N}(w)$ s.t. $\alpha \subseteq [B]$;
$w \Vdash \Diamond B$	iff	for all $\alpha \in \mathcal{N}(w)$, $\alpha \cap [B] \neq \emptyset$.

The neighbourhood semantics characterises the cube of classical non-normal modal logics:

Theorem 1 (Chellas [6]) *Logic* E(M,C,N) *is sound and complete with respect to neighbourhood models (which in addition are supplemented, closed under intersection, or contain the unit).*

3 Intuitionistic non-normal monomodal logics

Our definition of intuitionistic non-normal modal logics begins with monomodal logics, that is logics containing only one modality, either $\Box$ or $\Diamond$. We first define the axiomatic systems, and then present their sequent calculi.

Under "intuitionistic modal logics" we understand any modal logic L that extends intuitionistic propositional logic (IPL) and satisfies the following requirements:

(R1) L is conservative over IPL: its non-modal fragment coincides with IPL.
(R2) L satisfies the disjunction property: if $A \vee B$ is derivable, then at least one formula between A and B is also derivable.

3.1 Hilbert systems

From the point of view of axiomatic systems, two different classes of intuitionistic non-normal monomodal logics can be defined by analogy with the

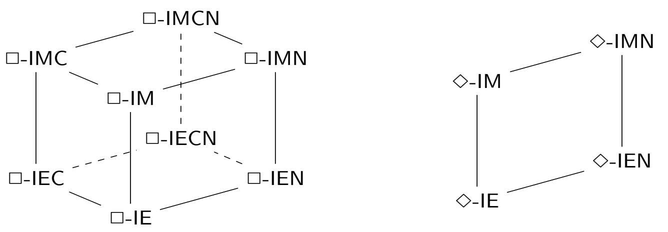

Fig. 3: The lattices of intuitionistic non-normal monomodal logics.

definition of classical non-normal modal logics (cf. Section 2). Intuitionistic modal logics are modal extensions of IPL, for which we consider the following axiomatisation:

$\supset$-1	$A \supset (B \supset A)$	$\wedge$-1	$A \wedge B \supset A$
$\supset$-2	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$	$\wedge$-2	$A \wedge B \supset B$
$\vee$-1	$A \supset A \vee B$	$\wedge$-3	$A \supset (B \supset A \wedge B)$
$\vee$-2	$B \supset A \vee B$	efq	$\bot \supset A$
$\vee$-3	$(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$	mp	$\dfrac{A \qquad A \supset B}{B}$

We define over IPL two families of intuitionistic non-normal monomodal logics, that depend on the considered modal operator, and are called therefore the □- and the ◇-family. The □-family is defined in language $\mathcal{L}_\Box := \mathcal{L} \setminus \{\Diamond\}$ by adding to IPL the rule $\mathsf{E}_\Box$ and any combination of axioms $\mathsf{M}_\Box$, $\mathsf{C}_\Box$ and $\mathsf{N}_\Box$. The ◇-family is defined instead in language $\mathcal{L}_\Diamond := \mathcal{L} \setminus \{\Box\}$ by adding to IPL the rule $\mathsf{E}_\Diamond$ and any combination of axioms $\mathsf{M}_\Diamond$ and $\mathsf{N}_\Diamond$. It is worth noticing that we do not consider intuitionistic non-normal modal logics containing axiom $\mathsf{C}_\Diamond$. We denote the resulting logics by, respectively, $\Box$-IE^* and $\Diamond$-IE^*, where E^* replaces any system of the classical cube (for ◇-logics, any system not containing $\mathsf{C}_\Diamond$).

Notice that, having rejected the definability of the lacking modality, □- and ◇-logics are distinct, as □ and ◇ behave differently. Moreover, as a consequence of the fact that the systems in the classical cube are pairwise non-equivalent, we have that the □-family contains eight distinct logics, while the ◇-family contains four distinct logics (something not derivable in a classical system is clearly not derivable in the corresponding intuitionistic system). It is also worth noticing that, as it happens in the classical case, axioms $\mathsf{M}_\Box$, $\mathsf{M}_\Diamond$ and $\mathsf{N}_\Box$ are interderivable, respectively, with rules $\mathsf{Mon}_\Box$, $\mathsf{Mon}_\Diamond$ and Nec, and that $\mathsf{K}_\Box$ is derivable from $\mathsf{M}_\Box$ and $\mathsf{C}_\Box$ (as the standard derivations are intuitionistically valid).

3.2 Sequent calculi

We now present sequent calculi for intuitionistic non-normal monomodal logics. The calculi are defined as modal extensions of a given sequent calculus for IPL. We take G3ip as the base calculus (Figure 4), and extend it with suitable

Ax $\Gamma, p \Rightarrow p$	L$\bot$ $\Gamma, \bot \Rightarrow A$
L$\wedge$ $\dfrac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}$	R$\wedge$ $\dfrac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$
L$\vee$ $\dfrac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C}$	R$\vee$ $\dfrac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1}$ $(i = 0, 1)$
L$\supset$ $\dfrac{\Gamma, A \supset B \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \supset B \Rightarrow C}$	R$\supset$ $\dfrac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$

Fig. 4: Rules of G3ip (Troelstra and Schwichtenberg [41]).

$\mathsf{E}_\Box^{\mathsf{seq}}$ $\dfrac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Box A \Rightarrow \Box B}$	$\mathsf{E}_\Diamond^{\mathsf{seq}}$ $\dfrac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Diamond A \Rightarrow \Diamond B}$
$\mathsf{M}_\Box^{\mathsf{seq}}$ $\dfrac{A \Rightarrow B}{\Gamma, \Box A \Rightarrow \Box B}$	$\mathsf{M}_\Diamond^{\mathsf{seq}}$ $\dfrac{A \Rightarrow B}{\Gamma, \Diamond A \Rightarrow \Diamond B}$
$\mathsf{N}_\Box^{\mathsf{seq}}$ $\dfrac{\Rightarrow A}{\Gamma \Rightarrow \Box A}$	$\mathsf{N}_\Diamond^{\mathsf{seq}}$ $\dfrac{A \Rightarrow}{\Gamma, \Diamond A \Rightarrow B}$
$\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$ $\dfrac{A_1, ..., A_n \Rightarrow B \qquad B \Rightarrow A_1 \ \ ... \ \ B \Rightarrow A_n}{\Gamma, \Box A_1, ..., \Box A_n \Rightarrow \Box B}$ $(n \geq 1)$	
$\mathsf{M}_\Box\mathsf{C}^{\mathsf{seq}}$ $\dfrac{A_1, ..., A_n \Rightarrow B}{\Gamma, \Box A_1, ..., \Box A_n \Rightarrow \Box B}$ $(n \geq 1)$	

Fig. 5: Modal rules for Gentzen calculi.

combinations of the modal rules in Figure 5. The $\Box$-rules can be compared with the rules given in Lavendhomme and Lucas [26], where sequent calculi for classical non-normal modal logics are presented. However, our rules are slightly different as (i) they have a single formula in the right-hand side of sequents; and (ii) contexts are added to the left-hand side of sequents appearing in the conclusion. Restriction (i) is adopted in order to have single-succedent calculi (as G3ip is), while with (ii) we implicitly embed weakening in the application of the modal rules. We consider the sequent calculi to be defined by the modal rules that are added to G3ip. The calculi are the following.

$$
\begin{array}{llllll}
\mathsf{G.\Box\text{-}IE} & := & \mathsf{E}^{\mathsf{seq}}_{\Box} & \mathsf{G.\Box\text{-}IEC} & := & \mathsf{E}_{\Box}\mathsf{C}^{\mathsf{seq}} \\
\mathsf{G.\Box\text{-}IM} & := & \mathsf{M}^{\mathsf{seq}}_{\Box} & \mathsf{G.\Box\text{-}IMC} & := & \mathsf{M}_{\Box}\mathsf{C}^{\mathsf{seq}} \\
\mathsf{G.\Box\text{-}IEN} & := & \mathsf{E}^{\mathsf{seq}}_{\Box} + \mathsf{N}^{\mathsf{seq}}_{\Box} & \mathsf{G.\Box\text{-}IECN} & := & \mathsf{E}_{\Box}\mathsf{C}^{\mathsf{seq}} + \mathsf{N}^{\mathsf{seq}}_{\Box} \\
\mathsf{G.\Box\text{-}IMN} & := & \mathsf{M}^{\mathsf{seq}}_{\Box} + \mathsf{N}^{\mathsf{seq}}_{\Box} & \mathsf{G.\Box\text{-}IMCN} & := & \mathsf{M}_{\Box}\mathsf{C}^{\mathsf{seq}} + \mathsf{N}^{\mathsf{seq}}_{\Box} \\
\mathsf{G.\Diamond\text{-}IE} & := & \mathsf{E}^{\mathsf{seq}}_{\Diamond} & & & \\
\mathsf{G.\Diamond\text{-}IM} & := & \mathsf{M}^{\mathsf{seq}}_{\Diamond} & & & \\
\mathsf{G.\Diamond\text{-}IEN} & := & \mathsf{E}^{\mathsf{seq}}_{\Diamond} + \mathsf{N}^{\mathsf{seq}}_{\Diamond} & & & \\
\mathsf{G.\Diamond\text{-}IMN} & := & \mathsf{M}^{\mathsf{seq}}_{\Diamond} + \mathsf{N}^{\mathsf{seq}}_{\Diamond} & & &
\end{array}
$$

Notice that — as in Lavendhomme and Lucas [26] — axiom $\mathsf{C}_{\Box}$ is captured by modifying the rules $\mathsf{E}^{\mathsf{seq}}_{\Box}$ and $\mathsf{M}^{\mathsf{seq}}_{\Box}$. In particular, these rules are replaced by $\mathsf{E}_{\Box}\mathsf{C}^{\mathsf{seq}}$ and $\mathsf{M}_{\Box}\mathsf{C}^{\mathsf{seq}}$, respectively, that are the generalisations of $\mathsf{E}^{\mathsf{seq}}_{\Box}$ and $\mathsf{M}^{\mathsf{seq}}_{\Box}$ with n principal formulas (instead of just one) in the left-hand side of sequents. Observe that $\mathsf{E}_{\Box}\mathsf{C}^{\mathsf{seq}}$ and $\mathsf{M}_{\Box}\mathsf{C}^{\mathsf{seq}}$ are non-standard, as they introduce an arbitrary number of modal formulas with a single application, and that $\mathsf{E}_{\Box}\mathsf{C}^{\mathsf{seq}}$ has in addition an arbitrary number of premisses. Another way of looking at $\mathsf{E}_{\Box}\mathsf{C}^{\mathsf{seq}}$ and $\mathsf{M}_{\Box}\mathsf{C}^{\mathsf{seq}}$ is to consider them as infinite sets of rules, each set containing a standard rule for any $n \geq 1$. Under the latter interpretation, the calculi are non-standard anyway, as they are defined by infinite sets of rules.

We now prove the admissibility of some structural rules, and then show the equivalence between the sequent calculi and the associated Hilbert systems.

Proposition 1 *The following weakening and contraction rules are height-preserving admissible in any monomodal calculus:*

$$
\mathsf{Lwk}\ \frac{\Gamma \Rightarrow B}{\Gamma, A \Rightarrow B} \qquad \mathsf{Rwk}\ \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A} \qquad \mathsf{ctr}\ \frac{\Gamma, A, A \Rightarrow B}{\Gamma, A \Rightarrow B}\ .
$$

Proof By induction on n, we show that whenever the premiss of an application of Lwk, Rwk or ctr has a derivation of height n, then its conclusion has a derivation of the same height. As usual, the proof considers the last rule applied in the derivation of the premiss (when the premiss is not an initial sequent). For left and right weakening, if the last rule applied is a rule of G3ip, then the proof is standard. If it is a modal rule, then the proof is easy. For instance, the premiss $\Gamma \Rightarrow$ of Rwk is necessarily derived by $\mathsf{N}^{\mathsf{seq}}_{\Diamond}$. Then Γ contains a formula $\Diamond B$ that is principal in the application of $\mathsf{N}^{\mathsf{seq}}_{\Diamond}$, which in turn has $B \Rightarrow$ as premiss. By a different application of $\mathsf{N}^{\mathsf{seq}}_{\Diamond}$ to $B \Rightarrow$ we can derive $\Gamma \Rightarrow A$ for any A.

For contraction, the proof is known if the last rule applied is a rule of G3ip. If this is a modal rule, then the proof is easy. The most interesting case is possibly when both occurrences of A in the premiss $\Gamma, A, A \Rightarrow B$ of ctr are principal in the last rule application. In this case, the last rule is either $\mathsf{E}_{\Box}\mathsf{C}^{\mathsf{seq}}$ or $\mathsf{M}_{\Box}\mathsf{C}^{\mathsf{seq}}$. If it is $\mathsf{M}_{\Box}\mathsf{C}^{\mathsf{seq}}$, then $A \equiv \Box C$ for some C, and the sequent is derived from $D_1, ..., D_n, C, C \Rightarrow$ for some $\Box D_1, ..., \Box D_n$ in Γ. By inductive hypothesis we can apply ctr to the last sequent and obtain $D_1, ..., D_n, C \Rightarrow$, and then by $\mathsf{M}_{\Box}\mathsf{C}^{\mathsf{seq}}$ derive sequent $\Gamma, A \Rightarrow B$, which is the conclusion of ctr (the proof is analogous for $\mathsf{E}_{\Box}\mathsf{C}^{\mathsf{seq}}$).

We now show that the cut rule

$$\mathsf{cut}\ \frac{\Gamma \Rightarrow A \qquad \Gamma', A \Rightarrow B}{\Gamma, \Gamma' \Rightarrow B}$$

is admissible in any monomodal calculus. The proof makes use of the following notion of weight of formulas:

Definition 2 (Weight of formulas) The function w assigning to each formula A its weight $w(A)$ is defined as follows: $w(\bot) = 0$; $w(p) = 1$; $w(A \circ B) = w(A) + w(B) + 1$ for $\circ \equiv \wedge, \vee, \supset$; and $w(\Box A) = w(\Diamond A) = w(A) + 2$.

Observe that, given the present definition, $\neg A$ has a smaller weight than $\Box A$ and $\Diamond A$. Although irrelevant to the next theorem, this shall be used in Section 4 for the proof of cut elimination for bimodal calculi.

Theorem 2 *The rule* cut *is admissible in any monomodal calculus.*

Proof Given a derivation of a sequent with some applications of cut, we show how to remove any such application and obtain a derivation of the same sequent without cut. The proof is by double induction, with primary induction on the weight of the cut formula and secondary induction on the cut height. We recall that, for any application of cut, the cut formula is the formula which is deleted by that application, while the cut height is the sum of the heights of the derivations of the premisses of cut.

We just consider the cases in which the cut formula is principal in the last rule applied in the derivation of both premisses of cut. Moreover, we treat explicitly only the cases in which both premisses are derived by modal rules, as the non-modal cases are already considered in the proof of cut admissibility for G3ip, and because modal and non-modal rules do not interact in any relevant way.

• ($\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$; $\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$). Let $\Gamma_1 = A_1, ..., A_n$ and $\Gamma_2 = C_1, ..., C_m$. The first derivation is converted into the second one, which contains several applications of cut on a cut formula of smaller weight.

$$\dfrac{\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}\ \dfrac{\Gamma_1 \Rightarrow B \quad B \Rightarrow A_1 \ \dots\ B \Rightarrow A_n}{\Gamma, \Box\Gamma_1, \Rightarrow \Box B} \qquad \dfrac{B, \Gamma_2 \Rightarrow D \quad D \Rightarrow B \quad D \Rightarrow C_1 \ \dots\ D \Rightarrow C_m}{\Gamma', \Box B, \Box\Gamma_2 \Rightarrow \Box D}\ \mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}}{\Gamma, \Gamma', \Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box D}\ \mathsf{cut}$$

$$\rightsquigarrow$$

$$\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}\ \dfrac{\mathsf{cut}\ \dfrac{\Gamma_1 \Rightarrow B \qquad B, \Gamma_2 \Rightarrow D}{\Gamma_1, \Gamma_2 \Rightarrow D} \qquad \left(\mathsf{cut}\ \dfrac{D \Rightarrow B \qquad B \Rightarrow A_i}{D \Rightarrow A_i}\right)_{i=1}^{n} \qquad D \Rightarrow C_1 \ \dots\ D \Rightarrow C_m}{\Gamma, \Gamma', \Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box D}$$

• ($\mathsf{M}_\Box\mathsf{C}^{\mathsf{seq}}$; $\mathsf{M}_\Box\mathsf{C}^{\mathsf{seq}}$) is analogous to ($\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$; $\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$). ($\mathsf{E}_\Box^{\mathsf{seq}}$; $\mathsf{E}_\Box^{\mathsf{seq}}$) and ($\mathsf{M}_\Box^{\mathsf{seq}}$; $\mathsf{M}_\Box^{\mathsf{seq}}$) are the particular cases where $n, m = 1$.

• ($\mathsf{N}_\Box^{\mathsf{seq}}$; $\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$). Let $\Gamma_1 = B_1, ..., B_n$. The first derivation is converted into the second one, which has an application of cut on a cut formula of smaller weight.

$$
\dfrac{\mathsf{N}^{\mathsf{seq}}_{\Box}\ \dfrac{\Rightarrow A}{\Gamma \Rightarrow \Box A} \qquad \dfrac{A, \Gamma_1 \Rightarrow C \quad C \Rightarrow A \quad C \Rightarrow B_1 \ \ldots\ C \Rightarrow B_n}{\Gamma', \Box A, \Box\Gamma_1 \Rightarrow \Box C}\ \mathsf{E}_{\Box}\mathsf{C}^{\mathsf{seq}}}{\Gamma, \Gamma', \Box\Gamma_1 \Rightarrow \Box C}\ \mathsf{cut}
$$

$$\rightsquigarrow$$

$$
\dfrac{\mathsf{cut}\ \dfrac{\Rightarrow A \qquad A, \Gamma_1 \Rightarrow C}{\Gamma_1 \Rightarrow C} \qquad C \Rightarrow B_1 \ \ldots\ C \Rightarrow B_n}{\Gamma, \Gamma', \Box\Gamma_1 \Rightarrow \Box C}\ \mathsf{E}_{\Box}\mathsf{C}^{\mathsf{seq}}
$$

• ($\mathsf{N}^{\mathsf{seq}}_{\Box}$; $\mathsf{M}_{\Box}\mathsf{C}^{\mathsf{seq}}$) is analogous to ($\mathsf{N}^{\mathsf{seq}}_{\Box}$; $\mathsf{E}_{\Box}\mathsf{C}^{\mathsf{seq}}$). ($\mathsf{N}^{\mathsf{seq}}_{\Box}$; $\mathsf{E}^{\mathsf{seq}}_{\Box}$) and ($\mathsf{N}^{\mathsf{seq}}_{\Box}$; $\mathsf{M}^{\mathsf{seq}}_{\Box}$) are the particular cases where $n = 1$.

• ($\mathsf{E}^{\mathsf{seq}}_{\Diamond}$; $\mathsf{E}^{\mathsf{seq}}_{\Diamond}$) and ($\mathsf{M}^{\mathsf{seq}}_{\Diamond}$; $\mathsf{M}^{\mathsf{seq}}_{\Diamond}$) are analogous to ($\mathsf{E}^{\mathsf{seq}}_{\Box}$; $\mathsf{E}^{\mathsf{seq}}_{\Box}$) and ($\mathsf{M}^{\mathsf{seq}}_{\Box}$; $\mathsf{M}^{\mathsf{seq}}_{\Box}$), respectively.

• ($\mathsf{E}^{\mathsf{seq}}_{\Diamond}$; $\mathsf{N}^{\mathsf{seq}}_{\Diamond}$).

$$
\dfrac{\mathsf{E}^{\mathsf{seq}}_{\Diamond}\ \dfrac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Diamond A \Rightarrow \Diamond B} \qquad \dfrac{B \Rightarrow}{\Gamma', \Diamond B \Rightarrow C}\ \mathsf{N}^{\mathsf{seq}}_{\Diamond}}{\Gamma, \Gamma', \Diamond A \Rightarrow C}\ \mathsf{cut}
\quad \rightsquigarrow \quad
\dfrac{\dfrac{A \Rightarrow B \qquad B \Rightarrow}{A \Rightarrow}\ \mathsf{cut}}{\Gamma, \Gamma', \Diamond A \Rightarrow C}\ \mathsf{N}^{\mathsf{seq}}_{\Diamond}
$$

• ($\mathsf{M}^{\mathsf{seq}}_{\Diamond}$; $\mathsf{N}^{\mathsf{seq}}_{\Diamond}$) is analogous to ($\mathsf{E}^{\mathsf{seq}}_{\Diamond}$; $\mathsf{N}^{\mathsf{seq}}_{\Diamond}$).

As a consequence of the admissibility of cut we obtain the equivalence between the sequent calculi and the axiomatic systems.

Theorem 3 *Let* L *be any intuitionistic non-normal monomodal logic. Then the calculus* $\mathsf{G.L}$ *is equivalent to system* L*.*

Proof We prove that the axioms and rules of L are derivable in $\mathsf{G.L}$. For the axioms of IPL we can consider their derivations in $\mathsf{G3ip}$, whereas mp is simulated by cut in the usual way. Here we show that any modal rule allows us to derive the corresponding axiom:

$$
\dfrac{\dfrac{\mathsf{cut}\ \dfrac{\Rightarrow A \supset B \qquad A, A \supset B \Rightarrow B}{A \Rightarrow B} \qquad \dfrac{\Rightarrow B \supset A \qquad B, B \supset A \Rightarrow A}{B \Rightarrow A}\ \mathsf{cut}}{\Box A \Rightarrow \Box B}\ \mathsf{E}^{\mathsf{seq}}_{\Box}}{\Rightarrow \Box A \supset \Box B}\ \mathsf{R}{\supset}
$$

$$
\dfrac{\dfrac{\dfrac{A, B \Rightarrow A \wedge B \qquad A \wedge B \Rightarrow A \qquad A \wedge B \Rightarrow B}{\Box A, \Box B \Rightarrow \Box(A \wedge B)}\ \mathsf{E}_{\Box}\mathsf{C}^{\mathsf{seq}}}{\Box A \wedge \Box B \Rightarrow \Box(A \wedge B)}\ \mathsf{L}{\wedge}}{\Rightarrow \Box A \wedge \Box B \supset \Box(A \wedge B)}\ \mathsf{R}{\supset}
\qquad
\dfrac{\Rightarrow \top}{\Rightarrow \Box\top}\ \mathsf{N}^{\mathsf{seq}}_{\Box}
$$

$$
\dfrac{\dfrac{\bot \Rightarrow}{\Diamond\bot \Rightarrow}\ \mathsf{N}^{\mathsf{seq}}_{\Diamond}}{\Rightarrow \neg\Diamond\bot}\ \mathsf{R}{\neg}
\qquad
\dfrac{\dfrac{A \wedge B \Rightarrow A}{\Box(A \wedge B) \Rightarrow \Box A}\ \mathsf{M}^{\mathsf{seq}}_{\Box}}{\Rightarrow \Box(A \wedge B) \supset \Box A}\ \mathsf{R}{\supset}
\qquad
\dfrac{\dfrac{A \Rightarrow A \vee B}{\Diamond A \Rightarrow \Diamond(A \vee B)}\ \mathsf{M}^{\mathsf{seq}}_{\Diamond}}{\Rightarrow \Diamond A \supset \Diamond(A \vee B)}\ \mathsf{R}{\supset}
$$

For the other direction, we prove that the rules of $\mathsf{G.L}$ are derivable in L. As before, it is enough to consider the modal rules. The derivations are in most cases straightforward, and we just consider the following ones, leaving the remaining cases to the reader:

- If L contains $\mathsf{N}_\Box$, then $\mathsf{N}_\Box^{\mathsf{seq}}$ is derivable. Assume $\vdash_{\mathsf{L}} A$. Then by Nec (which is equivalent to $\mathsf{N}_\Box$), $\vdash_{\mathsf{L}} \Box A$.
- If L contains $\mathsf{N}_\Diamond$, then $\mathsf{N}_\Diamond^{\mathsf{seq}}$ is derivable. Assume $\vdash_{\mathsf{L}} A \supset \bot$. Since $\vdash_{\mathsf{L}} \bot \supset A$, by $\mathsf{E}_\Diamond^{\mathsf{seq}}$, $\vdash_{\mathsf{L}} \Diamond A \supset \Diamond\bot$. Then $\vdash_{\mathsf{L}} \neg\Diamond\bot \supset \neg\Diamond A$, and, since $\vdash_{\mathsf{L}} \neg\Diamond\bot$, we have $\vdash_{\mathsf{L}} \neg\Diamond A$.
- If L contains $\mathsf{C}_\Box$, then $\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$ is derivable. Assume $\vdash_{\mathsf{L}} A_1 \wedge ... \wedge A_n \supset B$ and $\vdash_{\mathsf{L}} B \supset A_i$ for all $1 \leq i \leq n$. Then $\vdash_{\mathsf{L}} B \supset A_1 \wedge ... \wedge A_n$. By $\mathsf{E}_\Box$, $\vdash_{\mathsf{L}} \Box(A_1 \wedge ... \wedge A_n) \supset \Box B$. In addition, by several applications of $\mathsf{C}_\Box$, $\vdash_{\mathsf{L}} \Box A_1 \wedge ... \wedge \Box A_n \supset \Box(A_1 \wedge ... \wedge A_n)$. Therefore $\vdash_{\mathsf{L}} \Box A_1 \wedge ... \wedge \Box A_n \supset \Box B$.

4 Intuitionistic non-normal bimodal logics

In this section, we present intuitionistic non-normal modal logics endowed with two modalities $\Box$ and $\Diamond$. We first present their sequent calculi, and then give equivalent axiomatisations.

A simple way to define intuitionistic non-normal bimodal logics would be to consider the fusion of two monomodal logics that belong respectively to the $\Box$- and to the $\Diamond$-family. Given two logics $\Box$-IE^* and $\Diamond$-IE^*, their fusion in language $\mathcal{L}_\Box \cup \mathcal{L}_\Diamond$ is the smallest bimodal logics containing $\Box$-IE^* and $\Diamond$-IE^* (cf. Gabbay *et al.* [13] for the notion of fusion in normal modal logics; for the sake of simplicity we can assume that $\mathcal{L}_\Box$ and $\mathcal{L}_\Diamond$ share the same set of propositional variables, and differ only on $\Box$ and $\Diamond$). The resulting logic is axiomatised simply by adding to IPL the modal axioms and rules of $\Box$-IE^*, plus the modal axioms and rules of $\Diamond$-IE^*.

It is clear, however, that in the resulting systems the modalities do not interact at all, as there is no axiom involving both $\Box$ and $\Diamond$. On the contrary, finding suitable interactions between the modalities is often the main issue when intuitionistic bimodal logics are concerned. In that case, by reflecting the fact that, in IPL, the connectives are not interderivable, it is usually required that $\Box$ and $\Diamond$ are *not* dual. We take this lack of a duality as an additional requirement for the definition of intuitionistic non-normal bimodal logics:

(R3) $\Box$ and $\Diamond$ are not interdefinable.

In order to define intuitionistic non-normal bimodal logics by axiomatic systems, we would need to select the axioms between a plethora of possible formulas satisfying (R3). In the literature on intuitionistic normal modal logics many different axioms have been considered (see *e.g.* Fischer Servi [10], Wolter and Zakharyaschev [46], Simpson [38]). Here we take a different way, and define the logics starting with their sequent calculi.

We proceed as follows. Intuitionistic non-normal bimodal logics are defined by their sequent calculi. The calculi are conservative extensions of a given calculus for IPL, and have as modal rules some characteristic rules of intuitionistic

$$\mathsf{weak}_{\mathsf{a}}^{\mathsf{seq}} \; \frac{\Rightarrow A \qquad B \Rightarrow}{\Gamma, \Box A, \Diamond B \Rightarrow C} \qquad \mathsf{weak}_{\mathsf{b}}^{\mathsf{seq}} \; \frac{A \Rightarrow \qquad \Rightarrow B}{\Gamma, \Box A, \Diamond B \Rightarrow C}$$

$$\mathsf{neg}_{\mathsf{a}}^{\mathsf{seq}} \; \frac{A, B \Rightarrow \qquad \neg B \Rightarrow A}{\Gamma, \Box A, \Diamond B \Rightarrow C} \qquad \mathsf{neg}_{\mathsf{b}}^{\mathsf{seq}} \; \frac{A, B \Rightarrow \qquad \neg A \Rightarrow B}{\Gamma, \Box A, \Diamond B \Rightarrow C}$$

$$\mathsf{str}^{\mathsf{seq}} \; \frac{A, B \Rightarrow}{\Gamma, \Box A, \Diamond B \Rightarrow C}$$

Fig. 6: Interaction rules for sequent calculi.

non-normal monomodal logics, plus some rules connecting $\Box$ and $\Diamond$. In addition, we require that the cut rule is admissible. As usual, this means that adding rule cut to the calculus does not extend the set of derivable sequents.

To the purpose of defining the basic systems, we consider only interactions between $\Box$ and $\Diamond$ that can be seen as forms of "weak duality principles". In particular we consider the following question: For any two formulas A and B, under which conditions are $\Box A$ and $\Diamond B$ jointly inconsistent? We distinguish three degrees: $\Box A$ and $\Diamond B$ are jointly inconsistent when

(i) one of the two is $\top$ and the other is $\bot$.
(ii) A is equivalent to $\neg B$, or B is equivalent to $\neg A$.
(iii) A and B are jointly inconsistent.

Finally, we will distinguish logics that are monotonic from logics that are non-monotonic.

In practice, we realise this list of desiderata as follows. As before, we take G3ip (Figure 4) as the base calculus for intuitionistic logics. This is extended with combinations of the characteristic rules of intuitionistic non-normal monomodal logics given in Figure 5. The difference is that now the calculi contain both some rules for $\Box$ and some rules for $\Diamond$. In order to distinguish monotonic and non-monotonic logics, we require that the calculi contain either both $\mathsf{E}_{\Box}^{\mathsf{seq}}$ and $\mathsf{E}_{\Diamond}^{\mathsf{seq}}$ (in this case the corresponding logic will be non-monotonic), or both $\mathsf{M}_{\Box}^{\mathsf{seq}}$ and $\mathsf{M}_{\Diamond}^{\mathsf{seq}}$ (corresponding to monotonic logics). In addition, the calculi shall contain some of the interaction rules displayed in Figure 6. Since the logics are also distinguished with respect to the interaction between the modalities, we require that the calculi contain either both $\mathsf{weak}_{\mathsf{a}}^{\mathsf{seq}}$ and $\mathsf{weak}_{\mathsf{b}}^{\mathsf{seq}}$, or both $\mathsf{neg}_{\mathsf{a}}^{\mathsf{seq}}$ and $\mathsf{neg}_{\mathsf{b}}^{\mathsf{seq}}$, or $\mathsf{str}^{\mathsf{seq}}$.

In the following we present the sequent calculi for intuitionistic non-normal bimodal logics obtained by following this methodology. Then for each sequent calculus we define an equivalent axiomatisation. As we shall see, we end up with 24 distinct logics forming the lattice in Figure 9.

4.1 Sequent calculi for logics without $\mathsf{C}_\Box$

In the first part, we focus on sequent calculi for logics containing only axioms among $\mathsf{M}_\Box$, $\mathsf{M}_\Diamond$, $\mathsf{N}_\Box$ and $\mathsf{N}_\Diamond$ — in other words, we do not consider the axiom $\mathsf{C}_\Box$. The calculi are obtained by adding to $\mathsf{G3ip}$ (Figure 4) suitable combinations of the modal rules in Figures 5 and 6. Although in principle any combination of rules could define a calculus, we accept only the calculi that satisfy the restrictions mentioned above. In particular, this entails that we prove cut elimination. As usual, the first step towards the study of cut elimination is to prove the admissibility of the other structural rules.

Proposition 2 *Weakening and contraction are height-preserving admissible in each sequent calculus defined by any combination of modal rules in Figures 5 and 6.*

Proof The proposition is proved by extending the proof of Proposition 1 with an examination of the interaction rules in Figure 6. Due to their form, however, it is easy to verify that if the premiss of wk or ctr is derivable by any interaction rule, then the conclusion is derivable by the same rule.

We can now examine the admissibility of the cut rule. As stated by the following theorem, our methodology leads to consideration of 12 sequent calculi for intuitionistic non-normal bimodal logics.

Theorem 4 *The* cut *rule is admissible in the following calculi:*

$$\begin{array}{lll}
\mathsf{G.IE}_1 & := & \mathsf{E}_\Box^{\mathsf{seq}} + \mathsf{E}_\Diamond^{\mathsf{seq}} + \mathsf{weak}_{\mathsf{a}}^{\mathsf{seq}} + \mathsf{weak}_{\mathsf{b}}^{\mathsf{seq}} \\
\mathsf{G.IE}_2 & := & \mathsf{E}_\Box^{\mathsf{seq}} + \mathsf{E}_\Diamond^{\mathsf{seq}} + \mathsf{neg}_{\mathsf{a}}^{\mathsf{seq}} + \mathsf{neg}_{\mathsf{b}}^{\mathsf{seq}} \\
\mathsf{G.IE}_3 & := & \mathsf{E}_\Box^{\mathsf{seq}} + \mathsf{E}_\Diamond^{\mathsf{seq}} + \mathsf{str}^{\mathsf{seq}} \\
\mathsf{G.IM} & := & \mathsf{M}_\Box^{\mathsf{seq}} + \mathsf{M}_\Diamond^{\mathsf{seq}} + \mathsf{str}^{\mathsf{seq}}
\end{array}$$

Moreover, letting G^* *be any of the previous calculi,* cut *is admissible in*

$$\begin{array}{lll}
\mathsf{G}^*\mathsf{N}_\Diamond & := & \mathsf{G}^* + \mathsf{N}_\Diamond^{\mathsf{seq}} \\
\mathsf{G}^*\mathsf{N}_\Box & := & \mathsf{G}^* + \mathsf{N}_\Diamond^{\mathsf{seq}} + \mathsf{N}_\Box^{\mathsf{seq}}
\end{array}$$

Proof The structure of the proof is similar to the one of Theorem 2. As before, we consider only the cases where the cut formula is principal in the last rule applied in the derivation of both premisses, with the further restriction that the last rules are modal ones.

The combinations between $\Box$-rules, or between $\Diamond$-rules, have been already considered in the proof of Theorem 2. Therefore we only consider here the possible combinations of $\Box$- or $\Diamond$-rules with rules for interaction.

• ($\mathsf{E}_\Box^{\mathsf{seq}}$; $\mathsf{weak}_{\mathsf{a}}^{\mathsf{seq}}$). The derivation on the left is transformed into the derivation on the right.

$$\dfrac{\mathsf{E}_\Box^{\mathsf{seq}}\ \dfrac{A \Rightarrow B \quad B \Rightarrow A}{\Gamma, \Box A \Rightarrow \Box B} \qquad \dfrac{\Rightarrow B \quad C \Rightarrow}{\Gamma', \Box B, \Diamond C \Rightarrow D}\ \mathsf{weak}_{\mathsf{a}}^{\mathsf{seq}}}{\Gamma, \Gamma', \Box A, \Diamond C \Rightarrow D}\ \mathsf{cut} \quad \rightsquigarrow \quad \mathsf{weak}_{\mathsf{a}}^{\mathsf{seq}}\ \dfrac{\mathsf{cut}\ \dfrac{\Rightarrow B \quad B \Rightarrow A}{\Rightarrow A} \qquad C \Rightarrow}{\Gamma, \Gamma', \Box A, \Diamond C \Rightarrow D}$$

• ($\mathsf{E}_\Diamond^{\mathsf{seq}}$; $\mathsf{weak}_{\mathsf{a}}^{\mathsf{seq}}$).

$$\dfrac{\mathsf{E}^{\mathsf{seq}}_{\Diamond}\ \dfrac{A\Rightarrow B \quad B\Rightarrow A}{\Gamma,\Diamond A\Rightarrow\Diamond B} \qquad \dfrac{\Rightarrow C \quad B\Rightarrow}{\Gamma',\Box C,\Diamond B\Rightarrow D}\ \mathsf{weak}^{\mathsf{seq}}_{\mathsf{a}}}{\Gamma,\Gamma',\Diamond A,\Box C\Rightarrow D}\ \mathsf{cut} \quad\rightsquigarrow\quad \dfrac{\Rightarrow C \quad \dfrac{A\Rightarrow B \quad B\Rightarrow}{A\Rightarrow}\ \mathsf{cut}}{\Gamma,\Gamma',\Diamond A,\Box C\Rightarrow D}\ \mathsf{weak}^{\mathsf{seq}}_{\mathsf{a}}$$

- ($\mathsf{E}^{\mathsf{seq}}_{\Box}$; $\mathsf{weak}^{\mathsf{seq}}_{\mathsf{b}}$).

$$\dfrac{\mathsf{E}^{\mathsf{seq}}_{\Box}\ \dfrac{A\Rightarrow B \quad B\Rightarrow A}{\Gamma,\Box A\Rightarrow\Box B} \qquad \dfrac{B\Rightarrow \quad \Rightarrow C}{\Gamma',\Box B,\Diamond C\Rightarrow D}\ \mathsf{weak}^{\mathsf{seq}}_{\mathsf{b}}}{\Gamma,\Gamma',\Box A,\Diamond C\Rightarrow D}\ \mathsf{cut} \quad\rightsquigarrow\quad \mathsf{weak}^{\mathsf{seq}}_{\mathsf{b}}\ \dfrac{\mathsf{cut}\ \dfrac{A\Rightarrow B \quad B\Rightarrow}{A\Rightarrow} \quad \Rightarrow C}{\Gamma,\Gamma',\Box A,\Diamond C\Rightarrow D}$$

- ($\mathsf{E}^{\mathsf{seq}}_{\Diamond}$; $\mathsf{weak}^{\mathsf{seq}}_{\mathsf{b}}$).

$$\dfrac{\mathsf{E}^{\mathsf{seq}}_{\Diamond}\ \dfrac{A\Rightarrow B \quad B\Rightarrow A}{\Gamma,\Diamond A\Rightarrow\Diamond B} \qquad \dfrac{C\Rightarrow \quad \Rightarrow B}{\Gamma',\Box C,\Diamond B\Rightarrow D}\ \mathsf{weak}^{\mathsf{seq}}_{\mathsf{b}}}{\Gamma,\Gamma',\Diamond A,\Box C\Rightarrow D}\ \mathsf{cut} \quad\rightsquigarrow\quad \dfrac{\Rightarrow C \quad \dfrac{\Rightarrow B \quad B\Rightarrow A}{\Rightarrow A}\ \mathsf{cut}}{\Gamma,\Gamma',\Diamond A,\Box C\Rightarrow D}\ \mathsf{weak}^{\mathsf{seq}}_{\mathsf{b}}$$

- ($\mathsf{E}^{\mathsf{seq}}_{\Diamond}$; $\mathsf{neg}^{\mathsf{seq}}_{\mathsf{a}}$).

$$\dfrac{\mathsf{E}^{\mathsf{seq}}_{\Box}\ \dfrac{A\Rightarrow B \quad B\Rightarrow A}{\Gamma,\Diamond A\Rightarrow\Diamond B} \qquad \dfrac{C,B\Rightarrow \quad \neg B\Rightarrow C}{\Gamma',\Box C,\Diamond B\Rightarrow D}\ \mathsf{neg}^{\mathsf{seq}}_{\mathsf{b}}}{\Gamma,\Gamma',\Box C,\Diamond A\Rightarrow D}\ \mathsf{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\mathsf{cut}\ \dfrac{A\Rightarrow B \quad C,B\Rightarrow}{C,A\Rightarrow} \qquad \dfrac{\dfrac{B\Rightarrow A}{\neg A\Rightarrow\neg B} \quad \neg B\Rightarrow C}{\neg A\Rightarrow C}\ \mathsf{cut}}{\Gamma,\Gamma',\Box C,\Diamond A\Rightarrow D}\ \mathsf{neg}^{\mathsf{seq}}_{\mathsf{b}}$$

Observe that the second derivation has two applications of cut, both of them with a cut formula of smaller weight; in particular $w(\neg B) < w(\Diamond B)$ (cf. Definition 2).

- ($\mathsf{E}^{\mathsf{seq}}_{\Box}$; $\mathsf{neg}^{\mathsf{seq}}_{\mathsf{b}}$).

$$\dfrac{\mathsf{E}^{\mathsf{seq}}_{\Box}\ \dfrac{A\Rightarrow B \quad B\Rightarrow A}{\Gamma,\Box A\Rightarrow\Box B} \qquad \dfrac{B,C\Rightarrow \quad \neg B\Rightarrow C}{\Gamma',\Box B,\Diamond C\Rightarrow D}\ \mathsf{neg}^{\mathsf{seq}}_{\mathsf{b}}}{\Gamma,\Gamma',\Box A,\Diamond C\Rightarrow D}\ \mathsf{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\mathsf{cut}\ \dfrac{A\Rightarrow B \quad B,C\Rightarrow}{A,C\Rightarrow} \qquad \dfrac{\dfrac{B\Rightarrow A}{\neg A\Rightarrow\neg B} \quad \neg B\Rightarrow C}{\neg A\Rightarrow C}\ \mathsf{cut}}{\Gamma,\Gamma',\Box A,\Diamond C\Rightarrow D}\ \mathsf{neg}^{\mathsf{seq}}_{\mathsf{b}}$$

- ($\mathsf{M}^{\mathsf{seq}}_{\Box}$; $\mathsf{str}^{\mathsf{seq}}$).

$$\dfrac{\mathsf{M}^{\mathsf{seq}}_{\Box}\ \dfrac{A\Rightarrow B}{\Gamma,\Box A\Rightarrow\Box B} \qquad \dfrac{B,C\Rightarrow}{\Gamma',\Box B,\Diamond C\Rightarrow D}\ \mathsf{str}^{\mathsf{seq}}}{\Gamma,\Gamma',\Box A,\Diamond C\Rightarrow D}\ \mathsf{cut} \quad\rightsquigarrow\quad \dfrac{\dfrac{A\Rightarrow B \quad B,C\Rightarrow}{A,C\Rightarrow}\ \mathsf{cut}}{\Gamma,\Gamma',\Box A,\Diamond C\Rightarrow D}\ \mathsf{str}^{\mathsf{seq}}$$

- ($\mathsf{N}^{\mathsf{seq}}_{\Box}$; $\mathsf{str}^{\mathsf{seq}}$).

$$\dfrac{\mathsf{N}^{\mathsf{seq}}_{\Box}\ \dfrac{\Rightarrow A}{\Gamma\Rightarrow\Box A} \qquad \dfrac{A,B\Rightarrow}{\Gamma',\Box A,\Diamond B\Rightarrow C}\ \mathsf{str}^{\mathsf{seq}}}{\Gamma,\Gamma',\Diamond B\Rightarrow C}\ \mathsf{cut} \quad\rightsquigarrow\quad \dfrac{\dfrac{\Rightarrow A \quad A,B\Rightarrow}{B\Rightarrow}\ \mathsf{cut}}{\Gamma,\Gamma',\Diamond B\Rightarrow C}\ \mathsf{N}^{\mathsf{seq}}_{\Diamond}$$

The lacking combinations can be easily treated in similar ways. Observe that $\mathsf{N}_\Diamond^{\mathsf{seq}}$ does not interact significantly with any interaction rule, since the principal formula $\Diamond B$ occurs in the left-hand side of the concusion.

It can be shown that all combinations of rules excluded from Theorem 4 do not give a cut-free calculus. In particular, cut elimination fails if we take rule $\mathsf{N}_\Box^{\mathsf{seq}}$ and we do not take rule $\mathsf{N}_\Diamond^{\mathsf{seq}}$, or if we combine the monotonic rules for $\Box$ and $\Diamond$ with interaction rules different from $\mathsf{str}^{\mathsf{seq}}$. Here we show some explicative examples, we leave to the reader to check the remaining combinations.

Example 1 Sequent $\Diamond\bot \Rightarrow$ is derivable from $\mathsf{N}_\Box^{\mathsf{seq}}$ + $\mathsf{weak}_\mathsf{a}^{\mathsf{seq}}$ + cut (without $\mathsf{N}_\Diamond^{\mathsf{seq}}$), but it is not derivable from $\mathsf{N}_\Box^{\mathsf{seq}}$ + $\mathsf{weak}_\mathsf{a}^{\mathsf{seq}}$ without cut. A possible derivation is the following:

$$\dfrac{\mathsf{N}_\Box^{\mathsf{seq}}\ \dfrac{\Rightarrow \top}{\Rightarrow \Box\top} \qquad \dfrac{\Rightarrow \top \qquad \bot \Rightarrow}{\Box\top, \Diamond\bot \Rightarrow}\ \mathsf{weak}_\mathsf{a}^{\mathsf{seq}}}{\Diamond\bot \Rightarrow}\ \mathsf{cut}$$

It is clear however that sequent $\Diamond\bot \Rightarrow$ does not have any cut-free derivation without applying $\mathsf{N}_\Diamond^{\mathsf{seq}}$, as no rule different from $\mathsf{N}_\Diamond^{\mathsf{seq}}$ has $\Diamond\bot \Rightarrow$ in the conclusion. We shall consider in Section 7 a calculus containing $\mathsf{N}_\Box^{\mathsf{seq}}$ and not $\mathsf{N}_\Diamond^{\mathsf{seq}}$. As we shall see, that calculus has interaction rules of a different form.

Example 2 Sequent $\Box\neg p, \Diamond(p \wedge q) \Rightarrow$ is derivable from $\mathsf{M}_\Box^{\mathsf{seq}}$ + $\mathsf{neg}_\mathsf{a}^{\mathsf{seq}}$ + $\mathsf{neg}_\mathsf{b}^{\mathsf{seq}}$ + cut, but it is not derivable from $\mathsf{M}_\Box^{\mathsf{seq}}$ + $\mathsf{neg}_\mathsf{a}^{\mathsf{seq}}$ + $\mathsf{neg}_\mathsf{b}^{\mathsf{seq}}$ without cut. A possible derivation is as follows:

$$\dfrac{\mathsf{M}_\Box^{\mathsf{seq}}\ \dfrac{\neg p \Rightarrow \neg(p \wedge q)}{\Box\neg p \Rightarrow \Box\neg(p \wedge q)} \qquad \dfrac{\neg(p \wedge q), p \wedge q \Rightarrow \qquad \neg(p \wedge q) \Rightarrow \neg(p \wedge q)}{\Box\neg(p \wedge q), \Diamond(p \wedge q) \Rightarrow}\ \mathsf{neg}_\mathsf{a}^{\mathsf{seq}}}{\Box\neg p, \Diamond(p \wedge q) \Rightarrow}\ \mathsf{cut}$$

Let us now try to derive bottom-up the sequent without using cut. As a last rule we can only apply $\mathsf{neg}_\mathsf{a}^{\mathsf{seq}}$ or $\mathsf{neg}_\mathsf{b}^{\mathsf{seq}}$, as they are the only rules with a conclusion of the right form. In the first case, the premisses would be $\neg p, p \wedge q \Rightarrow$, and $\neg\neg p \Rightarrow p \wedge q$; while in the second case the premisses would be $\neg p, p \wedge q \Rightarrow$, and $\neg(p \wedge q) \Rightarrow \neg p$. It is clear, however, that in both cases the second premiss is not derivable.

4.2 Sequent calculi for logics with $\mathsf{C}_\Box$

We now consider sequent calculi for logics containing the axiom $\mathsf{C}_\Box$. We have seen in the case of monomodal $\Box$-logics that the rules for congruence and monotonicity of $\Box$ must be generalised to n principal boxed formulas in order to obtain cut-free calculi which capture $\mathsf{C}_\Box$. For the same reason, interaction rules need to be generalised in an analogous way. In this regard, observe that the rules in Figure 6 do not provide cut-free calculi if combined with $\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$ or $\mathsf{M}_\Box\mathsf{C}^{\mathsf{seq}}$, as the following example shows.

Example 3 The sequent $\Box p, \Box\neg p, \Diamond\top \Rightarrow$ is derivable by $\mathsf{M}_\Box\mathsf{C}^{\mathsf{seq}}$ + $\mathsf{weak}_\mathsf{a}^{\mathsf{seq}}$ + $\mathsf{weak}_\mathsf{b}^{\mathsf{seq}}$ + cut, but is not derivable by $\mathsf{M}_\Box\mathsf{C}^{\mathsf{seq}}$ + $\mathsf{weak}_\mathsf{a}^{\mathsf{seq}}$ + $\mathsf{weak}_\mathsf{b}^{\mathsf{seq}}$ without cut. The derivation with cut is as follows:

$$\dfrac{\mathsf{M}_\Box\mathsf{C}^{\mathsf{seq}}\ \dfrac{p, \neg p \Rightarrow \bot}{\Box p, \Box\neg p \Rightarrow \Box\bot} \qquad \dfrac{\bot \Rightarrow \qquad \Rightarrow \top}{\Box\bot, \Diamond\top \Rightarrow}\ \mathsf{weak}_\mathsf{b}^{\mathsf{seq}}}{\Box p, \Box\neg p, \Diamond\top \Rightarrow}\ \mathsf{cut}$$

In contrast, the sequent is not derivable without cut, as the only applicable rule would be $\mathsf{weak}_\mathsf{b}^{\mathsf{seq}}$, but neither p nor $\neg p$ is a contradiction.

Suitable generalisations of rules $\mathsf{weak}_\mathsf{b}^{\mathsf{seq}}$, $\mathsf{neg}_\mathsf{a}^{\mathsf{seq}}$, $\mathsf{str}^{\mathsf{seq}}$ are displayed in Figure 7. Observe that the rule $\mathsf{weak}_\mathsf{a}^{\mathsf{seq}}$ has not been modified, and more interestingly, that there is no rule corresponding to $\mathsf{neg}_\mathsf{b}^{\mathsf{seq}}$. Concerning $\mathsf{weak}_\mathsf{a}^{\mathsf{seq}}$, as a difference with other rules, the boxed formula which is principal in an application of $\mathsf{weak}_\mathsf{a}^{\mathsf{seq}}$ occurs unboxed only in the right-hand side of the premiss: for this reason the rule needs not to be modified (as shown in the proof of Theorem 5).

Concerning $\mathsf{neg}_\mathsf{b}^{\mathsf{seq}}$, its generalisation to n principal formulas would be as follows:

$$\mathsf{neg}_\mathsf{b}\mathsf{C}^{\mathsf{seq}}\ \dfrac{A_1, ..., A_n, B \Rightarrow \qquad \neg(A_1 \wedge ... \wedge A_n) \Rightarrow B}{\Gamma, \Box A_1, ..., \Box A_n, \Diamond B \Rightarrow C}\ .$$

This rule is not analytic as the right premiss contains a conjuction which does not occur in the conclusion. In contrast with the case of rules $\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$ and $\mathsf{neg}_\mathsf{a}^{\mathsf{seq}}$, it is not possible to decompose the right premiss into simpler premisses. In particular, notice that taking the n premisses $\neg A_1 \Rightarrow B, ..., \neg A_n \Rightarrow B$ is not the same as taking $\neg(A_1 \wedge ... \wedge A_n) \Rightarrow B$, since $\neg(A_1 \wedge ... \wedge A_n) \supset \neg A_1 \vee ... \vee \neg A_n$ is not valid in intuitionistic logic. At present it is an open problem whether adopting the rule $\mathsf{neg}_\mathsf{b}\mathsf{C}^{\mathsf{seq}}$ we would still obtain a cut-free calculus. For this reason we exclude this rule from the calculi for $\mathsf{C}_\Box$, and we stipulate that the calculi $\mathsf{G.IE}_2\mathsf{C}^*$ contain only rule $\mathsf{neg}_\mathsf{a}\mathsf{C}^{\mathsf{seq}}$. As a consequence, the calculi $\mathsf{G.IE}_2\mathsf{C}^*$ are not proper extensions of $\mathsf{G.IE}_2^*$.

As before, it can be easily proved that weakening and contraction are height-preserving admissible in the considered systems.

Proposition 3 *Weakening and contraction are height-preserving admissible in each sequent calculus defined by any combination of modal rules in Figures 5 and 7.*

Following our methodology, we obtain again 12 sequent calculi, as stated by the following theorem:

Theorem 5 *The rule* cut *is admissible in the following calculi:*

$$\begin{array}{lll}
\mathsf{G.IE}_1\mathsf{C} & := & \mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}} + \mathsf{E}_\Diamond^{\mathsf{seq}} + \mathsf{weak}_\mathsf{a}^{\mathsf{seq}} + \mathsf{weak}_\mathsf{b}\mathsf{C}^{\mathsf{seq}} \\
\mathsf{G.IE}_2\mathsf{C} & := & \mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}} + \mathsf{E}_\Diamond^{\mathsf{seq}} + \mathsf{neg}_\mathsf{a}\mathsf{C}^{\mathsf{seq}} \\
\mathsf{G.IE}_3\mathsf{C} & := & \mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}} + \mathsf{E}_\Diamond^{\mathsf{seq}} + \mathsf{strC}^{\mathsf{seq}} \\
\mathsf{G.IMC} & := & \mathsf{M}_\Box\mathsf{C}^{\mathsf{seq}} + \mathsf{M}_\Diamond^{\mathsf{seq}} + \mathsf{strC}^{\mathsf{seq}}
\end{array}$$

$$\mathsf{weak}_\mathsf{a}^\mathsf{seq}\ \frac{\Rightarrow A \qquad B \Rightarrow}{\Gamma, \Box A, \Diamond B \Rightarrow C} \qquad\qquad \mathsf{weak}_\mathsf{b}\mathsf{C}^\mathsf{seq}\ \frac{A_1, ..., A_n \Rightarrow \qquad \Rightarrow B}{\Gamma, \Box A_1, ..., \Box A_n, \Diamond B \Rightarrow C}$$

$$\mathsf{neg}_\mathsf{a}\mathsf{C}^\mathsf{seq}\ \frac{A_1, ..., A_n, B \Rightarrow \qquad \neg B \Rightarrow A_1 \ \ ... \ \ \neg B \Rightarrow A_n}{\Gamma, \Box A_1, ..., \Box A_n, \Diamond B \Rightarrow C}$$

$$\mathsf{strC}^\mathsf{seq}\ \frac{A_1, ..., A_n, B \Rightarrow}{\Gamma, \Box A_1, ..., \Box A_n, \Diamond B \Rightarrow C}$$

Fig. 7: Modified interaction rules for $\mathsf{C}_\Box$. For each rule we have $n \geq 1$.

Moreover, letting GC^* *be any of the previous calculi,* cut *is admissible in*

$$\mathsf{GC}^*\mathsf{N}_\Diamond \ := \ \mathsf{GC}^* + \mathsf{N}_\Diamond^\mathsf{seq}$$
$$\mathsf{GC}^*\mathsf{N}_\Box \ := \ \mathsf{GC}^* + \mathsf{N}_\Diamond^\mathsf{seq} + \mathsf{N}_\Box^\mathsf{seq}$$

Proof As before, we only present some relevant cases.

- ($\mathsf{E}_\Box\mathsf{C}^\mathsf{seq}$; $\mathsf{weak}_\mathsf{a}^\mathsf{seq}$). Let Γ_1 be the multiset $A_1, ..., A_n$, and $\Box\Gamma_1$ be $\Box A_1, ..., \Box A_n$.

$$\dfrac{\mathsf{E}_\Box\mathsf{C}^\mathsf{seq}\ \dfrac{\Gamma_1 \Rightarrow B \qquad B \Rightarrow A_1 \ \ ... \ \ B \Rightarrow A_n}{\Gamma, \Box\Gamma_1 \Rightarrow \Box B} \qquad \dfrac{\Rightarrow B \qquad C \Rightarrow}{\Gamma', \Box B, \Diamond C \Rightarrow D}\ \mathsf{weak}_\mathsf{a}^\mathsf{seq}}{\Gamma, \Gamma', \Box\Gamma_1, \Diamond C \Rightarrow D}\ \mathsf{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\mathsf{cut}\ \dfrac{\Rightarrow B \qquad B \Rightarrow A_1}{\Rightarrow A_1} \qquad C \Rightarrow}{\Gamma, \Gamma', \Box A_1, \Box A_2, ..., \Box A_n, \Diamond C \Rightarrow D}\ \mathsf{weak}_\mathsf{a}^\mathsf{seq}$$

- ($\mathsf{E}_\Box\mathsf{C}^\mathsf{seq}$; $\mathsf{neg}_\mathsf{a}\mathsf{C}^\mathsf{seq}$). Let $\Gamma_1 = A_1, ..., A_n$ and $\Gamma_2 = C_1, ..., C_m$.

$$\dfrac{\mathsf{E}_\Box\mathsf{C}^\mathsf{seq}\ \dfrac{\Gamma_1 \Rightarrow B \quad B \Rightarrow A_1 \ ... \ B \Rightarrow A_n}{\Gamma, \Box\Gamma_1 \Rightarrow \Box B} \qquad \dfrac{B, \Gamma_2, D \Rightarrow \quad \neg D \Rightarrow B \quad \neg D \Rightarrow C_1 \ ... \ \neg D \Rightarrow C_m}{\Gamma', \Box B, \Box\Gamma_2, \Diamond D \Rightarrow E}\ \mathsf{neg}_\mathsf{b}\mathsf{C}^\mathsf{seq}}{\Gamma, \Gamma', \Box\Gamma_1, \Box\Gamma_2, \Diamond D \Rightarrow E}\ \mathsf{cut}$$

$$\rightsquigarrow$$

$$\mathsf{E}_\Box\mathsf{C}^\mathsf{seq}\ \dfrac{\mathsf{cut}\ \dfrac{\Gamma_1 \Rightarrow B \qquad B, \Gamma_2, D \Rightarrow}{\Gamma_1, \Gamma_2, D \Rightarrow} \qquad \left(\mathsf{cut}\ \dfrac{\neg D \Rightarrow B \qquad B \Rightarrow A_i}{\neg D \Rightarrow A_i}\right)_{i=1}^{n} \qquad \neg D \Rightarrow C_1 \ ... \ \neg D \Rightarrow C_m}{\Gamma, \Gamma', \Box\Gamma_1, \Box\Gamma_2, \Diamond D \Rightarrow E}$$

- ($\mathsf{E}_\Box\mathsf{C}^\mathsf{seq}$; $\mathsf{strC}^\mathsf{seq}$). Let $\Gamma_1 = A_1, ..., A_n$ and $\Gamma_2 = C_1, ..., C_m$. We have:

$$\dfrac{\mathsf{E}_\Box\mathsf{C}^\mathsf{seq}\ \dfrac{\Gamma_1 \Rightarrow B \qquad B \Rightarrow A_1 \ \ ... \ \ B \Rightarrow A_n}{\Gamma, \Box\Gamma_1 \Rightarrow \Box B} \qquad \dfrac{B, \Gamma_2, D \Rightarrow}{\Gamma', \Box B, \Box\Gamma_2, \Diamond D \Rightarrow E}\ \mathsf{strC}^\mathsf{seq}}{\Gamma, \Gamma', \Box\Gamma_1, \Box\Gamma_2, \Diamond D \Rightarrow E}\ \mathsf{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow B \qquad B, \Gamma_2, D \Rightarrow}{\Gamma_1, \Gamma_2, D \Rightarrow}\ \mathsf{cut}}{\Gamma, \Box\Gamma_1, \Box\Gamma_2, \Diamond D \Rightarrow E}\ \mathsf{strC}^\mathsf{seq}$$

$\mathsf{weak_a}$	$\neg(\Box\top \wedge \Diamond\bot)$	$\mathsf{weak_b}$	$\neg(\Box\bot \wedge \Diamond\top)$	str	$\dfrac{\neg(A \wedge B)}{\neg(\Box A \wedge \Diamond B)}$
$\mathsf{neg_a}$	$\neg(\Box\neg A \wedge \Diamond A)$	$\mathsf{neg_b}$	$\neg(\Box A \wedge \Diamond\neg A)$		

Fig. 8: Hilbert axioms and rules for interactions between $\Box$ and $\Diamond$.

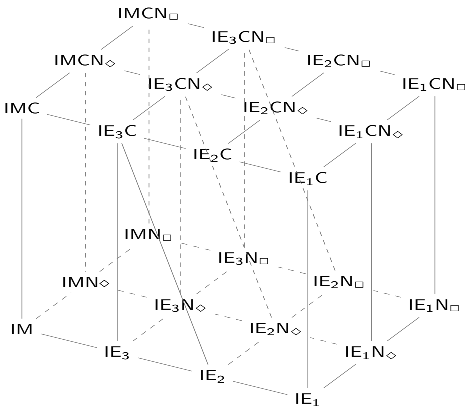

Fig. 9: The lattice of intuitionistic non-normal bimodal logics.

Notably, the cut-free calculi in Theorem 5 are the $\mathsf{C}_\Box$-versions of the cut-free calculi in Theorem 4, with the only exception of calculi $\mathsf{G.IE_2C}^*$ which do not contain any rule corresponding to $\mathsf{neg_b^{seq}}$. This means that, once the interaction rules are conveniently modified, the generalisation of the modal rules to n principal formulas preserves cut elimination.

4.3 Hilbert systems

For each sequent calculus, we now define an equivalent Hilbert system. To this purpose, in addition to the formulas of Figure 1, we also consider the axioms and rules displayed in Figure 8. As before, the Hilbert systems are defined by the set of modal axioms and rules that are added to IPL. The systems are axiomatised as follows:

$\mathsf{IE_1} := \mathsf{E}_\Box + \mathsf{E}_\Diamond + \mathsf{weak_a} + \mathsf{weak_b}$	$\mathsf{IE_1C} := \mathsf{IE_1} + \mathsf{C}_\Box$
$\mathsf{IE_2} := \mathsf{E}_\Box + \mathsf{E}_\Diamond + \mathsf{neg_a} + \mathsf{neg_b}$	$\mathsf{IE_2C} := \mathsf{E}_\Box + \mathsf{E}_\Diamond + \mathsf{neg_a} + \mathsf{C}_\Box$
$\mathsf{IE_3} := \mathsf{E}_\Box + \mathsf{E}_\Diamond + \mathsf{str}$	$\mathsf{IE_3C} := \mathsf{IE_3} + \mathsf{C}_\Box$
$\mathsf{IM} := \mathsf{E}_\Box + \mathsf{E}_\Diamond + \mathsf{M}_\Box + \mathsf{M}_\Diamond + \mathsf{str}$	$\mathsf{IMC} := \mathsf{IM} + \mathsf{C}_\Box$

Moreover, letting H^* be any of the four systems listed above, we consider the following additional systems:

$\mathsf{H^*N}_\Diamond := \mathsf{H}^* + \mathsf{N}_\Diamond$ $\qquad$ $\mathsf{H^*N}_\Box := \mathsf{H}^* + \mathsf{N}_\Box$

The different systems and their relations are depicted in Figure 9. Notice in particular that the systems $\mathsf{IE_2C}$, $\mathsf{IE_2CN}_\Diamond$ and $\mathsf{IE_2CN}_\Box$ are not extensions of, respectively, $\mathsf{IE_2}$, $\mathsf{IE_2N_\Diamond}$ and $\mathsf{IE_2N_\Box}$, as explained for the corresponding calculi on p. 18.

Theorem 6 *Let $G.L$ be any sequent calculus for intuitionistic non-normal bimodal logics. Then $\mathsf{G.L}$ is equivalent to system L.*

Proof We show that every axiom and rule of L is derivable in $\mathsf{G.L}$. We only consider here the interactions between the modalities, as the derivations of the other axioms have been already given in the proof of Theorem 3.

$$\dfrac{\dfrac{\dfrac{\Rightarrow \top \qquad \bot \Rightarrow}{\Box\top, \Diamond\bot \Rightarrow}\,\mathsf{weak}^{\mathsf{seq}}_{\mathsf{a}}}{\Box\top \wedge \Diamond\bot \Rightarrow}\,\mathsf{L}\wedge}{\Rightarrow \neg(\Box\top \wedge \Diamond\bot)}\,\mathsf{R}\neg \qquad \dfrac{\dfrac{\dfrac{\Rightarrow \top \qquad \bot \Rightarrow}{\Diamond\top, \Box\bot \Rightarrow}\,\mathsf{weak}^{\mathsf{seq}}_{\mathsf{b}}}{\Diamond\top \wedge \Box\bot \Rightarrow}\,\mathsf{L}\wedge}{\Rightarrow \neg(\Diamond\top \wedge \Box\bot)}\,\mathsf{R}\neg \qquad \dfrac{\dfrac{\dfrac{A, \neg A \Rightarrow \qquad \neg A \Rightarrow \neg A}{\Box\neg A, \Diamond A \Rightarrow}\,\mathsf{neg}^{\mathsf{seq}}_{\mathsf{a}}}{\Box\neg A \wedge \Diamond A \Rightarrow}\,\mathsf{L}\wedge}{\Rightarrow \neg(\Box\neg A \wedge \Diamond A)}\,\mathsf{R}\neg$$

$$\dfrac{\dfrac{\dfrac{A, \neg A \Rightarrow \qquad \neg A \Rightarrow \neg A}{\Box A, \Diamond\neg A \Rightarrow}\,\mathsf{neg}^{\mathsf{seq}}_{\mathsf{b}}}{\Box A \wedge \Diamond\neg A \Rightarrow}\,\mathsf{L}\wedge}{\Rightarrow \neg(\Box A \wedge \Diamond\neg A)}\,\mathsf{R}\neg \qquad \mathsf{cut}\,\dfrac{\Rightarrow \neg(A \wedge B) \qquad A, B, \neg(A \wedge B) \Rightarrow}{\dfrac{\dfrac{\dfrac{A, B \Rightarrow}{\Box A, \Diamond B \Rightarrow}\,\mathsf{str}^{\mathsf{seq}}}{\Box A \wedge \Diamond B \Rightarrow}\,\mathsf{L}\wedge}{\Rightarrow \neg(\Box A \wedge \Diamond B)}\,\mathsf{R}\neg}$$

Conversely, we prove that every rule of $\mathsf{G.L}$ is derivable in L. As before, we only need to consider the interaction rules. In most cases the derivations are easy to find, we give the following ones as examples.

- If L contains the axiom $\mathsf{weak_a}$, then the rule $\mathsf{weak}^{\mathsf{seq}}_{\mathsf{a}}$ is derivable. Assume that $\vdash_{\mathsf{L}} A$ and $\vdash_{\mathsf{L}} B \supset \bot$. Then $\vdash_{\mathsf{L}} \top \supset A$ and, since $\vdash_{\mathsf{L}} A \supset \top$, by $\mathsf{E}_\Box$ we have $\vdash_{\mathsf{L}} \Box A \supset \Box\top$. Moreover, since $\vdash_{\mathsf{L}} \bot \supset B$, by $\mathsf{E}_\Diamond$ we have $\vdash_{\mathsf{L}} \Diamond B \supset \Diamond\bot$, hence $\vdash_{\mathsf{L}} \neg\Diamond\bot \supset \neg\Diamond B$. By $\mathsf{weak_a}$ we also have $\vdash_{\mathsf{L}} \Box\top \supset \neg\Diamond\bot$. Thus $\vdash_{\mathsf{L}} \Box A \supset \neg\Diamond B$, which gives $\vdash_{\mathsf{L}} \neg(\Box A \wedge \Diamond B)$.
- If L contains the axiom $\mathsf{neg_b}$, then the rule $\mathsf{neg}^{\mathsf{seq}}_{\mathsf{b}}$ is derivable. Assume $\vdash_{\mathsf{L}} \neg(A \wedge B)$ – that is $\vdash_{\mathsf{L}} B \supset \neg A$ – and $\vdash_{\mathsf{L}} \neg B \supset A$. Then, by $\mathsf{E}_\Diamond$, $\vdash_{\mathsf{L}} \Diamond B \supset \Diamond\neg A$. By $\mathsf{neg_b}$ we have $\vdash_{\mathsf{L}} \Diamond\neg A \supset \neg\Box A$. Thus $\vdash_{\mathsf{L}} \Diamond B \supset \neg\Box A$, which gives $\vdash_{\mathsf{L}} \neg(\Box A \wedge \Diamond B)$.
- If L contains the axioms $\mathsf{C}_\Box$ and $\mathsf{neg_a}$, then the rule $\mathsf{neg_a C^{seq}}$ is derivable. Assume $\vdash_{\mathsf{L}} A_1 \wedge ... \wedge A_n \wedge B \supset \bot$ and $\vdash_{\mathsf{L}} \neg B \supset A_1$, ..., $\vdash_{\mathsf{L}} \neg B \supset A_n$. Then $\vdash_{\mathsf{L}} A_1 \wedge ... \wedge A_n \supset \neg B$ and $\vdash_{\mathsf{L}} \neg B \supset A_1 \wedge ... \wedge A_n$. By $\mathsf{E}_\Box$, $\vdash_{\mathsf{L}} \Box(A_1 \wedge ... \wedge A_n) \supset \Box\neg B$, and by considering axiom $\mathsf{C}_\Box$ $n-1$ times, $\vdash_{\mathsf{L}} \Box A_1 \wedge ... \wedge \Box A_n \supset \Box(A_1 \wedge ... \wedge A_n)$. Moreover, by $\mathsf{neg_a}$ we have $\vdash_{\mathsf{L}} \Box\neg B \supset \neg\Diamond B$. Then $\vdash_{\mathsf{L}} \Box A_1 \wedge ... \wedge \Box A_n \wedge \Diamond B \supset \bot$.

5 Decidability, and other consequences of cut elimination

Analytic cut-free sequent calculi are a very powerful tool for proof analysis. In this section, we take advantage of the admissibility of cut in all sequent calculi defined in Sections 3 and 4 in order to prove additional properties of the corresponding logics. Looking at the shape of the rules, we first observe that all calculi satisfy all requirements on intuitionistic non-normal modal logics that we have initially assumed, *i.e.* that they are conservative over IPL (R1); that they satisfy the disjunction property (R2); and that the duality principles $\mathsf{Dual}_\Box$ and $\mathsf{Dual}_\Diamond$ are not derivable (R3). In a similar way, we prove that all calculi are pairwise distinct, hence the lattices of intuitionistic non-normal modal logics contain, respectively, 8 distinct monomodal $\Box$-logics, 4 distinct monomodal $\Diamond$-logics, and 24 distinct bimodal logics.

Some form of subformula property often follows from cut elimination. By subformula property we mean as usual the property that given a root sequent $\Gamma \Rightarrow A$ of a derivation, every formula occurring in any sequent in any derivation of $\Gamma \Rightarrow A$ is a subformula of a formula in $\Gamma \Rightarrow A$. For calculi containing the rules $\mathsf{neg}_\mathsf{a}^\mathsf{seq}$ (or $\mathsf{neg}_\mathsf{a}\mathsf{C}^\mathsf{seq}$) and $\mathsf{neg}_\mathsf{b}^\mathsf{seq}$, we need to relax slightly the property by considering $\neg A$ as a "subformula" of $\Box A$ and $\Diamond A$. As we shall see, in all cases the subformula property is strong enough to provide, together with the admissibility of contraction, a standard proof of decidability for G3 calculi.

We conclude the section with some further remarks about the logics that we have defined, that in particular concern the relations between intuitionistic and classical modal logics.

Fact 1. Every intuitionistic non-normal modal logic defined in Section 3 and Section 4 satisfies the requirements R1, R2 and R3, the latter being only relevant for bimodal logics.

Proof (R1) Every logic is conservative over IPL: the non-modal rules of each sequent calculus are exactly the rules of G3ip.

(R2) Every logic satisfies the disjunction property: given a derivable sequent of the form $\Rightarrow A \lor B$, since no modal rule has such a conclusion, the last rule applied in its derivation is necessarily R$\lor$. This has premiss $\Rightarrow A$ or $\Rightarrow B$, which in turn is derivable.

(R3) For any system L, the axioms $\mathsf{Dual}_\Box$ and $\mathsf{Dual}_\Diamond$ are not derivable in L for an arbitrary A. In particular, neither $\neg\Box\neg p \supset \Diamond p$, nor $\neg\Diamond\neg p \supset \Box p$ (instances of the right-to-left implication of $\mathsf{Dual}_\Box$ and $\mathsf{Dual}_\Diamond$) is derivable. For instance, if we try to derive bottom-up the sequent $\neg\Box\neg p \Rightarrow \Diamond p$ in G.L, the only applicable rule would be L$\supset$. This has premiss $\neg\Box\neg p \Rightarrow \Box\neg p$. Again, L$\supset$ is the only applicable rule, with the same sequent as premiss (or, if contained by G.L, we could apply $\mathsf{N}_\Box^\mathsf{seq}$ to the non derivable sequent $\Rightarrow \neg p$). Since $\neg\Box\neg p \Rightarrow \Box\neg p$ is not an initial sequent, we have that $\neg\Box\neg p \Rightarrow \Diamond p$ is not derivable. The situation is analogous for $\neg\Diamond\neg p \Rightarrow \Box p$.

Theorem 7 *The lattice of intuitionistic non-normal bimodal logics contains 24* distinct *systems.*

Proof We leave to the reader to check that, given two logics L_1 and L_2 of the lattice, we can always find some formulas (or rules) that are derivable in L_1 and not in L_2, or *vice versa.* This can be easily done by considering the corresponding calculi $\mathsf{G.L}_1$ and $\mathsf{G.L}_2$. In particular, if L_1 is stronger than L_2, then the characteristic axiom of L_1 is not derivable in L_2. If instead L_1

and L_2 are incomparable, then they both have some characteristic axioms (or rules) that are not derivable in the other. For the rule str, we can consider the counterexample to cut elimination provided in Example 2.

Definition 3 (Strict subformula and negated subformula) For any formulas A and B, we say that A is a *strict subformula* of B if A is a subformula of B and $A \not\equiv B$. Moreover, we say that A is a *negated subformula* of B if there is a formula C such that C is a strict subformula of B and $A \equiv \neg C$.

Definition 4 (Subformula property and negated subformula property) We say that a sequent calculus G.L enjoys the *subformula property* if all formulas in any derivation are subformulas of the endsequent. We say that G.L enjoys the *negated subformula property* if all formulas in any derivation are either subformulas or negated subformulas of the endsequent.

The following result is an immediate consequence of cut elimination:

Theorem 8 *Any sequent calculus different from* $\mathsf{G.IE_2(C,N_\Diamond,N_\Box)}$ *enjoys the subformula property. Moreover, calculi* $\mathsf{G.IE_2(C,N_\Diamond,N_\Box)}$ *enjoy the negated subformula property.*

Given that the calculi enjoy the subformula property, we can extend to our logics the proof of decidability for G3ip given in Troelstra and Schwichtenberg [41] and thereby obtain a proof of decidability for our calculi. Schematically, the argument is as follows: all rules are analytical, and the premisses of each rule — with the exception of L$\supset$ — have a smaller complexity than the conclusion. In the same way of [41], the application of L$\supset$ can be controlled by an easy loop-checking. It turns out that given a root sequent $\Gamma \Rightarrow A$, every derivation of $\Gamma \Rightarrow A$ is finite and there are only a finite number of possible derivations of it. This implies decidability: the decision procedure consists trivially in checking all possible derivations.

Theorem 9 (Decidability) *For any intuitionistic non-normal modal logic defined in Section 3 and Section 4, it is decidable whether a given formula is derivable.*

We conclude this section with some remarks about the logics we have defined. Notice that there are three different systems — that is $\mathsf{IE_1}$, $\mathsf{IE_2}$, $\mathsf{IE_3}$ — that could be considered as counterparts of the same classical logic — that is logic E — and the same holds for some of their extensions. This is essentially due to the loss of duality between $\Box$ and $\Diamond$, that allows us to consider interactions of different strengths that are equally derivable in classical logic but are not intuitionistically equivalent. It is normally expected that an intuitionistic modal logic is strictly weaker than the corresponding classical modal logic, essentially because IPL is weaker than CPL. In this respect it is not true that $\mathsf{IE_3}$ is a counterpart of classical E. Indeed, the rule str is classically equivalent to $\mathsf{Mon_\Box}$, and hence not derivable in E. At the same time, however, it would be unnatural to consider $\mathsf{IE_3}$ as corresponding to classical M, as neither $\mathsf{M_\Box}$ nor $\mathsf{M_\Diamond}$ is derivable.

We see therefore that the picture of systems that emerge from a certain set of logic principles is richer in the intuitionistic case than in the classical one. The case of $\mathsf{IE_3}$ also suggests that assuming an intuitionistic base not only allows us to make subtle distinctions between principles that are not distinguishable in classical logic, but also gives us the possibility to investigate systems that in a sense lie between two different classical logics, and do not correspond essentially to any of the two.

6 Semantics

In this section, we present a semantics for all systems defined in Sections 3 and 4. As we shall see, our semantics represents a general framework for intuitionistic modal logics, that is able to capture modularly further intuitionistic non-normal modal logics such as CK and $\mathsf{CCDL^p}$. Models are obtained by combining intuitionistic Kripke models and neighbourhood models — see Definition 1 — in the following way:

Definition 5 A *Coupled Intuitionistic Neighbourhood Model* (CINM) is a tuple $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set, $\preceq$ is a preorder over $\mathcal{W}$, $\mathcal{V}$ is a hereditary valuation function $\mathcal{W} \longrightarrow \mathcal{P}(Atm)$ (*i.e.* $w \preceq v$ implies $\mathcal{V}(w) \subseteq \mathcal{V}(v)$), and $\mathcal{N}_\Box, \mathcal{N}_\Diamond$ are two neighbourhood functions $\mathcal{W} \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}))$ such that:

$$w \preceq v \text{ implies } \mathcal{N}_\Box(w) \subseteq \mathcal{N}_\Box(v) \text{ and } \mathcal{N}_\Diamond(w) \supseteq \mathcal{N}_\Diamond(v) \quad (hp).$$

Functions $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ can be *supplemented*, *closed under intersection*, or *contain the unit* — cf. properties in Definition 1. Moreover, let us define

$$-\alpha = \{w \in \mathcal{W} \mid \text{for all } v \succeq w, v \notin \alpha\}.$$

Then $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ can be related in the following ways:

For all $w \in \mathcal{W}$, $\mathcal{N}_\Box(w) \subseteq \mathcal{N}_\Diamond(w)$	Weak interaction ($weakInt$);
If $-\alpha \in \mathcal{N}_\Box(w)$, then $\mathcal{W} \setminus \alpha \in \mathcal{N}_\Diamond(w)$	Negation closure int_a ($negInt_a$);
If $\alpha \in \mathcal{N}_\Box(w)$, then $\mathcal{W} \setminus -\alpha \in \mathcal{N}_\Diamond(w)$	Negation closure int_b ($negInt_b$);
If $\alpha \in \mathcal{N}_\Box(w)$ and $\alpha \subseteq \beta$, then $\beta \in \mathcal{N}_\Diamond(w)$	Strong interaction ($strInt$).

The forcing relation $w \Vdash A$ associated to CINMs is defined as follows:

$w \Vdash p$	iff	$p \in \mathcal{V}(w)$;
$w \nVdash \bot$;		
$w \Vdash B \wedge C$	iff	$w \Vdash A$ and $w \Vdash B$;
$w \Vdash B \vee C$	iff	$w \Vdash A$ or $w \Vdash B$;
$w \Vdash B \supset C$	iff	for all $v \succeq w$, $v \Vdash B$ implies $v \Vdash C$;
$w \Vdash \Box B$	iff	$[B] \in \mathcal{N}_\Box(w)$;
$w \Vdash \Diamond B$	iff	$\mathcal{W} \setminus [B] \notin \mathcal{N}_\Diamond(w)$.

CINMs for monomodal logics $\Box$-IE^* and $\Diamond$-IE^* are defined by removing, respectively, $\mathcal{N}_\Diamond$ or $\mathcal{N}_\Box$ from the above definition (as well as the forcing condition for the lacking modality), and are called $\Box$-INMs and $\Diamond$-INMs.

As usual, given a class $\mathcal{C}$ of CINMs, we say that a formula A is *satisfiable* in $\mathcal{C}$ if there are $\mathcal{M} \in \mathcal{C}$ and $w \in \mathcal{M}$ such that $w \Vdash A$, and that A is *valid* in $\mathcal{C}$ if for all $\mathcal{M} \in \mathcal{C}$ and $w \in \mathcal{M}$ we have that $w \Vdash A$.

Observe that we are taking for $\supset$ the satisfaction clause of intuitionistic Kripke models, while for $\Box$ and $\Diamond$ we are taking the satisfaction clauses of classical neighbourhood models. In contrast with classical neighbourhood models, however, our models are endowed with two neighbourhood functions $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ rather than one. In this way, we can consider different relations between the two functions as given in the previous definition. Intuitively, these conditions correspond to the interaction axioms and rules. Interestingly, in the context of classical deontic logic, a neighbourhood semantics with separate neighbourhood functions for the two deontic modalities has been considered in Anglberger *et al.* [1]. A different neighbourhood semantics for an intuitionistic non-normal modal logic is defined in Kojima [24]. We shall discuss Kojima's semantics in Section 7.

The way functions $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ are related to the order $\preceq$ by the condition (hp) guarantees that CINMs preserve the hereditary property of intuitionistic Kripke models:

Proposition 4 *CINMs satisfy the hereditary property: for all $A \in \mathcal{L}$, if $w \Vdash A$ and $w \preceq v$, then $v \Vdash A$.*

Proof By induction on A. For the non-modal cases the proof is standard. For $A \equiv \Box B, \Diamond B$ it is immediate by (hp).

Depending on its axioms, to each system are associated models with specific properties, as summarised in the following table:

$\mathsf{M}_\Box$	$\mathcal{N}_\Box$ is supplemented
$\mathsf{N}_\Box$	$\mathcal{N}_\Box$ contains the unit
$\mathsf{C}_\Box$	$\mathcal{N}_\Box$ is closed under $\cap$
$\mathsf{M}_\Diamond$	$\mathcal{N}_\Diamond$ is supplemented
$\mathsf{N}_\Diamond$	$\mathcal{N}_\Diamond$ contains the unit

$\mathsf{weak_a} + \mathsf{weak_b}$	$weakInt$
$\mathsf{neg_a}$	$negInt_a$
$\mathsf{neg_b}$	$negInt_b$
str	$strInt$

In case of supplemented models — that is, when both $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ are supplemented — it suffices to consider $weakInt$ as the semantic condition corresponding to any interaction axiom (or rule). Indeed, it is easy to verify that whenever a model $\mathcal{M}$ satisfies $weakInt$, and $\mathcal{N}_\Box$ or $\mathcal{N}_\Diamond$ is supplemented, then $\mathcal{M}$ also satisfies $negInt_a$, $negInt_b$, and $strInt$.

Given the semantic properties of the above table, we have that $\Box$-INMs coincide essentially with the neighbourhood spaces by Goldblatt [17], although in that work the property of containing the unit is not considered. The only difference is that in Goldblatt's spaces the neighbourhoods are assumed to be closed with respect to the order, that is:

$$\text{If } \alpha \in \mathcal{N}_\Box(w),\ v \in \alpha \text{ and } v \preceq u, \text{ then } u \in \alpha \qquad (\preceq\text{-closure}).$$

As already observed by Goldblatt, however, this property is irrelevant from the point of view of the validity of formulas, as a formula A is valid in $\Box$-INMs (that are supplemented, closed under intersection, contain the unit) if and only if it is valid in the corresponding $\Box$-INMs that satisfy also the $\preceq$-closure. It is easy to verify that the same equivalence holds for CINMs for bimodal logics when considering the $\preceq$-closure only for the neighbourhoods in $\mathcal{N}_\Box$ (and not for those in $\mathcal{N}_\Diamond$).

The soundness of intuitionistic non-normal modal logics with respect to the corresponding CINMs can be proved easily.

Theorem 10 (Soundness) *Every intuitionistic non-normal modal logic is sound with respect to the corresponding CINMs.*

Proof It is easy to prove that a given axiom is valid whenever the corresponding property is satisfied. For $\mathsf{neg_a}$ and $\mathsf{neg_b}$ notice that $-[A] = [\neg A]$.

We now prove completeness of the Hilbert systems, whence of the sequent calculi, by the canonical model construction. In the following, let L be any intuitionistic non-normal modal logic and $\mathcal{L}$ be the corresponding language. We call L-*prime* any set X of formulas of $\mathcal{L}$ which is consistent ($X \not\vdash_\mathsf{L} \bot$), closed under derivation ($X \vdash_\mathsf{L} A$ implies $A \in X$) and such that if $(A \vee B) \in X$, then $A \in X$ or $B \in X$. For all $A \in \mathcal{L}$, we denote with $\uparrow_{pr} A$ the class of prime sets X such that $A \in X$. The standard properties of prime sets hold, in particular:

Lemma 1 (a) *If $X \not\vdash_\mathsf{L} A \supset B$, then there is a L-prime set Y such that $X \cup \{A\} \subseteq Y$ and $B \notin Y$.*

(b) *For any $A, B \in \mathcal{L}$, $\uparrow_{pr} A \subseteq \uparrow_{pr} B$ implies $\vdash_\mathsf{L} A \supset B$.*

As in the classical case (cf. Chellas [6]), for the proof of completeness we need to consider separately monotonic and non-monotonic systems. We first consider canonical models for non-monotonic systems, then we define canonical models enjoying supplementation for monotonic ones.

Definition 6 (Canonical models for non-monotonic systems) Let L be any system not containing axioms $\mathsf{M}_\Box$ and $\mathsf{M}_\Diamond$. The *canonical model* $\mathcal{M}^c$ for L is defined as the tuple $\langle \mathcal{W}^c, \preceq^c, \mathcal{N}_\Box^c, \mathcal{N}_\Diamond^c, \mathcal{V}^c \rangle$, where:

- $\mathcal{W}^c$ is the class of L-prime sets;
- for all $X, Y \in \mathcal{W}^c$, $X \preceq^c Y$ if and only if $X \subseteq Y$;
- $\mathcal{N}_\Box^c(X) = \{\uparrow_{pr} A \mid \Box A \in X\}$;
- $\mathcal{N}_\Diamond^c(X) = \mathcal{P}(\mathcal{W}^c) \setminus \{\mathcal{W}^c \setminus \uparrow_{pr} A \mid \Diamond A \in X\}$;
- $\mathcal{V}^c(X) = \{p \in \mathcal{L} \mid p \in X\}$.

Notice that the canonical model $\mathcal{M}^c$ is well defined, in particular it follows from the definition that $X \preceq^c Y$ implies both $\mathcal{N}_\Box^c(X) \subseteq \mathcal{N}_\Box^c(Y)$ and $\mathcal{N}_\Diamond^c(X) \supseteq \mathcal{N}_\Diamond^c(Y)$. We prove the following lemma.

Lemma 2 *Let L be any non-monotonic system and $\mathcal{M}^c = \langle \mathcal{W}^c, \preceq^c, \mathcal{N}_\Box^c, \mathcal{N}_\Diamond^c, \mathcal{V}^c \rangle$ be the canonical model for L. Then for all $X \in \mathcal{W}^c$ and all $A \in \mathcal{L}$ we have*

$$X \Vdash A \quad \text{iff} \quad A \in X.$$

Moreover: (i) If L *contains* $\mathsf{N}_\Box$*, then* $\mathcal{N}_\Box^c$ *contains the unit;*

(ii) If L *contains* $\mathsf{C}_\Box$*, then* $\mathcal{N}_\Box^c$ *is closed under intersection;*

(iii) If L *contains* $\mathsf{N}_\Diamond$*, then* $\mathcal{N}_\Diamond^c$ *contains the unit;*

(iv) If L *contains* $\mathsf{weak_a}$ *and* $\mathsf{weak_b}$*, then* $\mathcal{M}^c$ *is weakInt;*

(v) If L *contains* $\mathsf{neg_a}$*, then* $\mathcal{M}^c$ *is* $negInt_a$*;*

(vi) If L *contains* $\mathsf{neg_b}$*, then* $\mathcal{M}^c$ *is* $negInt_b$*;*

(vii) If L *contains* str*, then* $\mathcal{M}^c$ *is strInt.*

Proof By induction on A we prove that $X \Vdash A$ if and only if $A \in X$. If $A \equiv p,\ \bot,\ B \wedge C,\ B \vee C$, or $B \supset C$, the proof is immediate.

If $A \equiv \Box B$: for the converse implication, assume $\Box B \in X$. Then by definition $\uparrow_{pr} B \in \mathcal{N}_\Box^c(X)$, and by inductive hypothesis, $\uparrow_{pr} B = [B]_{\mathcal{M}^c}$, therefore $X \Vdash \Box B$. For the direct implication, assume $X \Vdash \Box B$. Then we have $[B]_{\mathcal{M}^c} \in \mathcal{N}_\Box^c(X)$, and, by inductive hypothesis, $[B]_{\mathcal{M}^c} = \uparrow_{pr} B$. By definition, this means that there is $C \in \mathcal{L}$ such that $\Box C \in X$ and $\uparrow_{pr} C = \uparrow_{pr} B$. Then, by Lemma 1, $\vdash_{\mathsf{L}} C \supset B$ and $\vdash_{\mathsf{L}} B \supset C$. Thus by $\mathsf{E}_\Box$, $\vdash_{\mathsf{L}} \Box C \supset \Box B$, and, by closure under derivation, $\Box B \in X$.

If $A \equiv \Diamond B$: for the converse implication, assume $\Diamond B \in X$. Then by definition $\mathcal{W}^c \setminus \uparrow_{pr} B \notin \mathcal{N}_\Diamond^c(X)$, and by inductive hypothesis, $\uparrow_{pr} B = [B]_{\mathcal{M}^c}$, therefore $X \Vdash \Diamond B$. For the direct implication, assume $X \Vdash \Diamond B$. Then we have $\mathcal{W}^c \setminus [B]_{\mathcal{M}^c} \notin \mathcal{N}_\Diamond^c(X)$, and, by inductive hypothesis, $\mathcal{W}^c \setminus \uparrow_{pr} B \notin \mathcal{N}_\Diamond^c(X)$. This means that there is $C \in \mathcal{L}$ such that $\Diamond C \in X$ and $\uparrow_{pr} C = \uparrow_{pr} B$. Thus, $\vdash_{\mathsf{L}} C \supset B$ and $\vdash_{\mathsf{L}} B \supset C$, therefore by $\mathsf{E}_\Diamond$, $\vdash_{\mathsf{L}} \Diamond C \supset \Diamond B$. By closure under derivation, we obtain that $\Diamond B \in X$.

Claims (i)–(vii) are proved as follows: (i) $\Box\top \in X$ for every $X \in \mathcal{W}^c$. Then by definition $\mathcal{W}^c = \uparrow_{pr} \top \in \mathcal{N}_\Box^c(X)$.

(ii) Assume $\alpha, \beta \in \mathcal{N}^c(X)$. Then there are $A, B \in \mathcal{L}$ such that $\Box A, \Box B \in X$, $\alpha = \uparrow_{pr} A$ and $\beta = \uparrow_{pr} B$. By closure under derivation, $\Box(A \wedge B) \in X$, and, by definition, $\uparrow_{pr}(A \wedge B) \in \mathcal{N}_\Box^c(X)$, where $\uparrow_{pr}(A \wedge B) = \uparrow_{pr} A \cap \uparrow_{pr} B = \alpha \cap \beta$.

(iii) $\neg\Diamond\bot \in X$ for every $X \in \mathcal{W}^c$, thus by consistency, $\Diamond\bot \notin X$. If $\mathcal{W}^c \setminus \uparrow_{pr} \bot \notin \mathcal{N}_\Box^c(X)$, then there is $A \in \mathcal{L}$ such that $\uparrow_{pr} A = \uparrow_{pr} \bot$ and $\Diamond A \in X$, that implies $\Diamond\bot \in X$. Therefore $\mathcal{W}^c = \mathcal{W}^c \setminus \uparrow_{pr} \bot \in \mathcal{N}_\Box^c(X)$.

(iv) Assume by contradiction that $\alpha \in \mathcal{N}_\Box^c(X)$ and $\alpha \notin \mathcal{N}_\Diamond^c(X)$. Then there are $A, B \in \mathcal{L}$ such that $\alpha = \uparrow_{pr} A$, $\alpha = \mathcal{W}^c \setminus \uparrow_{pr} B$, and $\Box A, \Diamond B \in X$, therefore $\uparrow_{pr} A = \mathcal{W}^c \setminus \uparrow_{pr} B$. By the properties of prime sets, this implies that $\vdash_{\mathsf{L}} \neg(A \wedge B)$ and $\vdash_{\mathsf{L}} A \vee B$, and by the disjunction property, $\vdash_{\mathsf{L}} A$ or $\vdash_{\mathsf{L}} B$. If we assume $\vdash_{\mathsf{L}} A$, then $\vdash_{\mathsf{L}} A \supset\subset \top$ and $\vdash_{\mathsf{L}} B \supset\subset \bot$. Therefore by $\mathsf{E}_\Box$ and $\mathsf{E}_\Diamond$, $\vdash_{\mathsf{L}} \Box A \supset \Box\top$ and $\vdash_{\mathsf{L}} \Diamond B \supset \Diamond\bot$, thus by closure under derivation, $\Box\top, \Diamond\bot \in X$. But $\neg(\Box\top \wedge \Diamond\bot) \in X$, in contradiction with the consistency of prime sets. If we now assume $\vdash_{\mathsf{L}} B$, then $\vdash_{\mathsf{L}} B \supset\subset \top$ and $\vdash_{\mathsf{L}} A \supset\subset \bot$. We obtain an analogous contradiction considering $\neg(\Diamond\top \wedge \Box\bot)$.

(v) By contraposition, assume that $\mathcal{W}^c \setminus \alpha \notin \mathcal{N}_\Diamond^c(X)$. Then there is $A \in \mathcal{L}$ such that $\mathcal{W}^c \setminus \alpha = \mathcal{W}^c \setminus \uparrow_{pr} A$ and $\Diamond A \in X$. Thus $\alpha = \uparrow_{pr} A$, and by $\mathsf{neg_a}$, $\Box\neg A \notin X$. Therefore $\uparrow_{pr} \neg A \notin \mathcal{N}_\Box^c(X)$ — otherwise there would be $\Box B \in X$

such that $\uparrow_{pr}\neg A = \uparrow_{pr} B$, which implies $\Box\neg A \in X$. Since $\uparrow_{pr}\neg A = - \uparrow_{pr} A = -\alpha$, the claim holds.

(vi) Assume $\alpha \in \mathcal{N}^c_\Box(X)$. Then there is $A \in \mathcal{L}$ such that $\alpha = \uparrow_{pr} A$ and $\Box A \in X$. Thus, by $\mathsf{neg_b}$ and consistency of X, $\Diamond\neg A \notin X$. Therefore $\mathcal{W}^c\setminus \uparrow_{pr}\neg A \in \mathcal{N}^c_\Diamond(X)$ (otherwise there would be $B \in \mathcal{L}$ such that $\uparrow_{pr} B = \uparrow_{pr}\neg A$ and $\Diamond B \in X$, which implies $\Diamond\neg A \in X$). Since $\uparrow_{pr}\neg A = - \uparrow_{pr} A$ ($\uparrow_{pr}\neg A = [\neg A]_{\mathcal{M}^c} = -[A]_{\mathcal{M}^c} = - \uparrow_{pr} A$) and $- \uparrow_{pr} A = -\alpha$, the claim holds.

(vii) Assume by contradiction that $\alpha \in \mathcal{N}^c_\Box(X)$, $\alpha \subseteq \beta$, and $\beta \notin \mathcal{N}^c_\Diamond(X)$. Then there are $A, B \in \mathcal{L}$ such that $\alpha = \uparrow_{pr} A$, $\beta = \mathcal{W}^c\setminus \uparrow_{pr} B$ and $\Box A, \Diamond B \in X$. Moreover, $\uparrow_{pr} A \subseteq \mathcal{W}^c\setminus \uparrow_{pr} B$, which implies $\uparrow_{pr} A \cap \uparrow_{pr} B = \emptyset$. Thus $\vdash_{\mathsf{L}} \neg(A \land B)$; and by str we have $\vdash_{\mathsf{L}} \neg(\Box A \land \Diamond B)$, in contradiction with the consistency of X.

We now define canonical models and prove an analogous lemma for monotonic systems. We shorten the proof by considering, instead of $\mathsf{M}_\Box$ and $\mathsf{M}_\Diamond$, the syntactically equivalent rules $\mathsf{Mon}_\Box$ and $\mathsf{Mon}_\Diamond$.

Definition 7 (Canonical models for monotonic systems) Let L be any system containing axioms $\mathsf{M}_\Box$ and $\mathsf{M}_\Diamond$. The *canonical model* $\mathcal{M}^c_+$ for L is the tuple $\langle \mathcal{W}^c, \preceq^c, \mathcal{N}^+_\Box, \mathcal{N}^+_\Diamond, \mathcal{V}^c\rangle$, where $\mathcal{W}^c, \preceq^c, \mathcal{V}^c$ are defined as in Definition 6, and:

$\mathcal{N}^+_\Box(X) = \{\alpha \subseteq \mathcal{W}^c \mid \text{there is } A \in \mathcal{L} \text{ s.t. } \Box A \in X \text{ and } \uparrow_{pr} A \subseteq \alpha\}$;

$\mathcal{N}^+_\Diamond(X) = \mathcal{P}(\mathcal{W}^c) \setminus \{\alpha \subseteq \mathcal{W}^c \mid \text{there is } A \in \mathcal{L} \text{ s.t. } \Diamond A \in X \text{ and } \alpha \subseteq \mathcal{W}^c\setminus \uparrow_{pr} A\}$.

Lemma 3 *Let L be any monotonic system and $\mathcal{M}^c_+ = \langle \mathcal{W}^c, \preceq^c, \mathcal{N}^+_\Box, \mathcal{N}^+_\Diamond, \mathcal{V}^c\rangle$ be the canonical model for L. Then $X \Vdash A$ if and only if $A \in X$. Moreover, claims (i)–(iii) of Lemma 2 still hold. Finally: (iv), if L contains* str*, then $\mathcal{M}^c_+$ is weakInt.*

Proof Observe that both $\mathcal{N}^+_\Box$ and $\mathcal{N}^+_\Diamond$ are supplemented. The proof is by induction on A, we only show the modal cases.

If $A \equiv \Box B$: for the converse implication, assume that $\Box B \in X$. Then by definition $\uparrow_{pr} B \in \mathcal{N}^+_\Box(X)$, and by inductive hypothesis, $\uparrow_{pr} B = [B]_{\mathcal{M}^c_+}$, therefore $X \Vdash \Box B$. For the direct implication, assume that $X \Vdash \Box B$. Then we have $[B]_{\mathcal{M}^c_+} \in \mathcal{N}^+_\Box(X)$, and, by inductive hypothesis, $[B]_{\mathcal{M}^c_+} = \uparrow_{pr} B$. By definition, this means that there is $C \in \mathcal{L}$ such that $\Box C \in X$ and $\uparrow_{pr} C \subseteq \uparrow_{pr} B$, which then implies $\vdash_{\mathsf{L}} C \supset B$. Thus, by $\mathsf{Mon}_\Box$, $\vdash_{\mathsf{L}} \Box C \supset \Box B$, and, by closure under derivation, $\Box B \in X$.

If $A \equiv \Diamond B$: for the converse implication, assume that $\Diamond B \in X$. Then by definition $\mathcal{W}^c\setminus \uparrow_{pr} B \notin \mathcal{N}^+_\Diamond(X)$, and by inductive hypothesis, $\uparrow_{pr} B = [B]_{\mathcal{M}^c_+}$, therefore $X \Vdash \Diamond B$. For the direct implication, assume $X \Vdash \Diamond B$. Then we have $\mathcal{W}^c \setminus [B]_{\mathcal{M}^c} \notin \mathcal{N}^+_\Diamond(X)$, and, by inductive hypothesis, $\mathcal{W}^c\setminus \uparrow_{pr} B \notin \mathcal{N}^+_\Diamond(X)$. This means that there is $C \in \mathcal{L}$ such that $\Diamond C \in X$ and $\mathcal{W}^c \setminus B \subseteq \mathcal{W}^c \setminus C$, that is $\uparrow_{pr} C \subseteq \uparrow_{pr} B$. Thus, $\vdash_{\mathsf{L}} C \supset B$, therefore by $\mathsf{E}_\Diamond$, $\vdash_{\mathsf{L}} \Diamond C \supset \Diamond B$. By closure under derivation we then have $\Diamond B \in X$.

Claims (i)–(iii) are proved similarly to the claims (i)–(iii) in Lemma 2. For (iv): By contradiction, assume that $\alpha \in \mathcal{N}_\Box^+(X)$ and $\alpha \notin \mathcal{N}_\Diamond^+(X)$. Then there are $A, B \in \mathcal{L}$ such that $\uparrow_{pr} A \subseteq \alpha$, $\alpha \subseteq \mathcal{W}^c \setminus \uparrow_{pr} B$, and $\Box A, \Diamond B \in X$. Therefore $\uparrow_{pr} A \subseteq \mathcal{W}^c \setminus \uparrow_{pr} B$, which implies $\vdash_\mathsf{L} \neg(A \wedge B)$. By str we then have $\neg(\Box A \wedge \Diamond B) \in X$, in contradiction with the consistency of X.

Theorem 11 (Completeness) *Every intuitionistic non-normal bimodal logic is complete with respect to the corresponding CINMs.*

Proof Assume that $\not\vdash_\mathsf{L} A$. Then $\not\vdash_\mathsf{L} \top \supset A$, thus, by Lemma 1, there is an L-prime set X such that $A \notin X$. By definition, X belongs to the canonical model $\mathcal{M}^c$ for L (resp. $\mathcal{M}^c_+$ for monotonic logics), and by Lemma 2, $\mathcal{M}^c, X \not\Vdash A$ ($\mathcal{M}^c_+, X \not\Vdash A$). By the properties of $\mathcal{M}^c$ ($\mathcal{M}^c_+$) we obtain completeness of L with respect to the corresponding models.

It can be easily verified that by removing $\mathcal{N}_\Diamond^c$ (resp. $\mathcal{N}_\Diamond^+$) or $\mathcal{N}_\Box^c$ (resp. $\mathcal{N}_\Box^+$) from the definition of $\mathcal{M}^c$ (resp. $\mathcal{M}^c_+$), we obtain analogous results for monomodal logics. Therefore we have:

Theorem 12 *Every intuitionistic non-normal monomodal logic is complete with respect to the corresponding CINMs.*

6.1 Finite model property and decidability

We have seen that all intuitionistic non-normal modal logics defined in Section 3 and 4 are sound and complete with respect to a certain class of CINMs. Here we prove that all logics have the finite model property (FMP), meaning that if a formula is satisfiable, then it has a finite model. Since finite models can be enumerated, FMP implies that the logics are decidable, thus providing a semantic proof of decidability which is alternative to the syntactic one presented in Section 5. As usual, FMP is based on the filtration technique. Given a model, this technique allows us to define a finite model which is equivalent to the initial one with respect to a finite set of formulas. The proofs are given explicitly for bimodal logics, while the simpler proofs for monomodal logics can be easily extracted.

Given a CINM $\mathcal{M}$ and a set Φ of formulas of $\mathcal{L}$ that is closed under subformulas, we define the equivalence relation $\sim$ on $\mathcal{W}$ as follows:

$$w \sim v \quad \text{iff} \quad \text{for all } A \in \Phi,\ w \Vdash A \text{ iff } v \Vdash A.$$

For any $w \in \mathcal{W}$ and $\alpha \subseteq \mathcal{W}$, we denote with $w_\sim$ the equivalence class containing w, and with $\alpha_\sim$ the set $\{w_\sim \mid w \in \alpha\}$ (thus in particular $[A]^\sim_\mathcal{M}$ is the set $\{w_\sim \mid w \in [A]_\mathcal{M}\}$).

Definition 8 Let $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ be a CINM and Φ be a set of formulas of $\mathcal{L}$ closed under subformulas. A *filtration* of $\mathcal{M}$ through Φ (or Φ-filtration) is any CINM $\mathcal{M}^* = \langle \mathcal{W}^*, \preceq^*, \mathcal{N}_\Box^*, \mathcal{N}_\Diamond^*, \mathcal{V}^* \rangle$ such that:

- $\mathcal{W}^* = \{w_\sim \mid w \in \mathcal{W}\}$;
- $w_\sim \preceq^* v_\sim$ iff for all $A \in \Phi$, $\mathcal{M}, w \Vdash A$ implies $\mathcal{M}, v \Vdash A$;
- for all $\Box A \in \Phi$, $[A]^\sim_\mathcal{M} \in \mathcal{N}^*_\Box(w_\sim)$ iff $[A]_\mathcal{M} \in \mathcal{N}_\Box(w)$;
- for all $\Diamond A \in \Phi$, $\mathcal{W}^* \setminus [A]^\sim_\mathcal{M} \in \mathcal{N}^*_\Diamond(w_\sim)$ iff $\mathcal{W} \setminus [A]_\mathcal{M} \in \mathcal{N}_\Diamond(w)$;
- for all $p \in \Phi$, $p \in \mathcal{V}^*(w_\sim)$ iff $p \in \mathcal{V}(w)$.

Observe that the model $\mathcal{M}^*$ is well-defined: for all $\Box A, \Diamond B, p \in \Phi$ we have that $w \sim v$ implies: (i) $[A]^\sim_\mathcal{M} \in \mathcal{N}^*_\Box(w_\sim)$ iff $[A]^\sim_\mathcal{M} \in \mathcal{N}^*_\Box(v_\sim)$; (ii) $\mathcal{W}^* \setminus [B]^\sim_\mathcal{M} \in \mathcal{N}^*_\Diamond(w_\sim)$ iff $\mathcal{W}^* \setminus [B]^\sim_\mathcal{M} \in \mathcal{N}^*_\Diamond(v_\sim)$; and (iii) $p \in \mathcal{V}^*(w_\sim)$ iff $p \in \mathcal{V}^*(v_\sim)$. Moreover, the conditions of CINMs are respected: (iv) $\preceq^*$ is a preorder; (v) $\mathcal{V}^*$ is hereditary; (vi) if $w_\sim \preceq^* v_\sim$ and $\Box A \in \Phi$, then $[A]^\sim_\mathcal{M} \in \mathcal{N}^*_\Box(w_\sim)$ implies $[A]^\sim_\mathcal{M} \in \mathcal{N}^*_\Box(v_\sim)$; (vii) if $w_\sim \preceq^* v_\sim$ and $\Diamond B \in \Phi$, then $\mathcal{W}^* \setminus [B]^\sim_\mathcal{M} \in \mathcal{N}^*_\Diamond(v_\sim)$ implies $\mathcal{W}^* \setminus [B]^\sim_\mathcal{M} \in \mathcal{N}^*_\Diamond(w_\sim)$; and (viii) for all $\alpha \subseteq \mathcal{W}^*$, $\alpha \in \mathcal{N}^*_\Box(w_\sim)$ implies $\alpha \in \mathcal{N}^*_\Diamond(w_\sim)$.

Lemma 4 (Filtration lemma) *For every formula $A \in \Phi$,*

$$\mathcal{M}^*, w_\sim \Vdash A \text{ iff } \mathcal{M}, w \Vdash A.$$

Proof Notice that it is equivalent to prove that $[A]_{\mathcal{M}^*} = [A]^\sim_\mathcal{M}$. The proof is by induction on A. For $A \equiv p,\ \bot,\ B \wedge C$, or $B \vee C$, the proof is immediate.

$A \equiv B \supset C$. Assume $\mathcal{M}, w \nVdash B \supset C$. Then there is $v \succeq w$ such that $\mathcal{M}, v \Vdash B$ and $\mathcal{M}, v \nVdash C$. By inductive hypothesis $\mathcal{M}^*, v_\sim \Vdash B$ and $\mathcal{M}^*, v_\sim \nVdash C$. Moreover, by definition of $\preceq^*$ and the fact that $\mathcal{M}$ satisfies the hereditary property, $w_\sim \preceq^* v_\sim$. Therefore $\mathcal{M}^*, w_\sim \nVdash B \supset C$. Now assume $\mathcal{M}^*, w_\sim \nVdash B \supset C$. Then there is $v_\sim \in \mathcal{W}^*$ such that $w_\sim \preceq^* v_\sim$, $\mathcal{M}^*, v_\sim \Vdash B$ and $\mathcal{M}^*, v_\sim \nVdash C$. By inductive hypothesis $\mathcal{M}, v \Vdash B$ and $\mathcal{M}, v \nVdash C$, thus $\mathcal{M}, v \nVdash B \supset C$. By definition of $\preceq^*$ we then have $\mathcal{M}, w \nVdash B \supset C$.

$A \equiv \Box B$. $\mathcal{M}^*, w_\sim \Vdash \Box B$ iff $[B]_{\mathcal{M}^*} \in \mathcal{N}^*_\Box(w_\sim)$ iff (i.h.) $[B]^\sim_\mathcal{M} \in \mathcal{N}^*_\Box(w_\sim)$ iff $[B]_\mathcal{M} \in \mathcal{N}_\Box(w)$ iff $\mathcal{M}, w \Vdash \Box B$.

$A \equiv \Diamond B$. $\mathcal{M}^*, w_\sim \Vdash \Diamond B$ iff $\mathcal{W}^* \setminus [B]_{\mathcal{M}^*} \notin \mathcal{N}^*_\Diamond(w_\sim)$ iff (i.h.) $\mathcal{W}^* \setminus [B]^\sim_\mathcal{M} \notin \mathcal{N}^*_\Diamond(w_\sim)$ iff $\mathcal{W} \setminus [B]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$ iff $\mathcal{M}, w \Vdash \Diamond B$.

Lemma 5 *Let $\mathcal{M}^*$ be a Φ-filtration of $\mathcal{M}$. (i) If $\mathcal{N}_\Box$ contains the unit and $\Box\top \in \Phi$, then $\mathcal{N}^*_\Box$ contains the unit. (ii) If $\mathcal{N}_\Diamond$ contains the unit and $\Diamond\bot \in \Phi$, then $\mathcal{N}^*_\Diamond$ contains the unit.*

Proof The claims follow from Definition 8 and Lemma 4, for instance if $\mathcal{W} = [\top]_\mathcal{M} \in \mathcal{N}_\Box(w)$, then, since $\Box\top \in \Phi$, we have $[\top]^\sim_\mathcal{M} = [\top]_{\mathcal{M}^*} = \mathcal{W}^* \in \mathcal{N}^*_\Box(w_\sim)$.

Whereas filtrations are sufficient for some basic models, for the other cases we must consider a finer notion that, following Chellas [6], we call *finest* filtration.

Definition 9 We call *finest* Φ-filtration any Φ-filtration $\mathcal{M}^*$ of $\mathcal{M}$ such that:

$\mathcal{N}^*_\Box(w_\sim) = \{[A]^\sim_\mathcal{M} \mid \Box A \in \Phi \text{ and } [A]_\mathcal{M} \in \mathcal{N}_\Box(w)\}$; and
$\mathcal{N}^*_\Diamond(w_\sim) = \mathcal{P}(\mathcal{W}^*) \setminus \{\mathcal{W}^* \setminus [A]^\sim_\mathcal{M} \mid \Diamond A \in \Phi \text{ and } \mathcal{W} \setminus [A]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)\}$.

Moreover, let $\mathcal{M}^\circ = \langle \mathcal{W}^*, \preceq^*, \mathcal{N}_\Box^\circ, \mathcal{N}_\Diamond^\circ, \mathcal{V}^* \rangle$ be a CINM where $\mathcal{W}^*$, $\preceq^*$ and $\mathcal{V}^*$ are as in $\mathcal{M}^*$. We say that:

- $\mathcal{M}^\circ$ is the *supplementation* of $\mathcal{M}^*$ if:

  $\alpha \in \mathcal{N}_\Box^\circ(w_\sim)$ iff there is $\beta \in \mathcal{N}_\Box^*(w_\sim)$ s.t. $\beta \subseteq \alpha$;
  $\alpha \notin \mathcal{N}_\Diamond^\circ(w_\sim)$ iff there is $\beta \notin \mathcal{N}_\Diamond^*(w_\sim)$ s.t. $\alpha \subseteq \beta$.
- $\mathcal{M}^\circ$ is the *intersection closure* of $\mathcal{M}^*$ if $\mathcal{N}_\Diamond^\circ(w_\sim) = \mathcal{N}_\Diamond^*(w_\sim)$, and

  $\alpha \in \mathcal{N}_\Box^\circ(w_\sim)$ iff there are $\alpha_1, ..., \alpha_n \in \mathcal{N}_\Box^*(w_\sim)$ s.t. $\alpha_1 \cap ... \cap \alpha_n = \alpha$.
- $\mathcal{M}^\circ$ is the *quasi-filtering* of $\mathcal{M}^*$ if:

  $\alpha \in \mathcal{N}_\Box^\circ(w_\sim)$ iff there are $\alpha_1, ..., \alpha_n \in \mathcal{N}_\Box^*(w_\sim)$ s.t. $\alpha_1 \cap ... \cap \alpha_n \subseteq \alpha$;
  $\alpha \notin \mathcal{N}_\Diamond^\circ(w_\sim)$ iff there is $\beta \notin \mathcal{N}_\Diamond^*(w_\sim)$ s.t. $\alpha \subseteq \beta$.

It is easy to verify that the supplementation of a model $\mathcal{M}$ is supplemented, its intersection closure is closed under intersection, and its quasi-filtering is both supplemented and closed under intersection.

Lemma 6 *Let $\mathcal{M}^*$ be a finest Φ-filtration of $\mathcal{M}$. (i) If $\mathcal{M}$ is weakInt, then $\mathcal{M}^*$ is weakInt. (ii) If $\mathcal{M}$ is strInt, then $\mathcal{M}^*$ is strInt. (iii) If $\mathcal{M}$ is $negInt_a$ and Φ is such that $\neg A \in \Phi$ for all $\Diamond A \in \Phi$, then $\mathcal{M}^*$ is $negInt_a$. (iv) If $\mathcal{M}$ is $negInt_b$ and Φ is such that $\neg A \in \Phi$ for all $\Box A \in \Phi$, then $\mathcal{M}^*$ is $negInt_b$.*

Proof (i) Assume by contradiction that $\alpha \in \mathcal{N}_\Box^*(w_\sim)$ and $\alpha \notin \mathcal{N}_\Diamond^*(w_\sim)$. Then $\alpha = [A]_\mathcal{M}^\sim$ for a $A \in \mathcal{L}$ such that $\Box A \in \Phi$ and $[A]_\mathcal{M} \in \mathcal{N}_\Box(w)$. Moreover $\alpha = \mathcal{W}^* \setminus [B]_\mathcal{M}^\sim$ for a $B \in \mathcal{L}$ such that $\Diamond B \in \Phi$ and $\mathcal{W} \setminus [B]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$. Thus $[A]_\mathcal{M}^\sim = \mathcal{W}^* \setminus [B]_\mathcal{M}^\sim$, which implies $[A]_\mathcal{M} = \mathcal{W} \setminus [B]_\mathcal{M}$ ($w \in [A]_\mathcal{M}$ iff $w_\sim \in [A]_\mathcal{M}^\sim$ iff $w_\sim \in \mathcal{W}^* \setminus [B]_\mathcal{M}^\sim$ iff $w \in \mathcal{W} \setminus [B]_\mathcal{M}$). Then, since $\mathcal{M}$ is *weakInt*, $\mathcal{W} \setminus [B]_\mathcal{M} \in \mathcal{N}_\Diamond(w)$, which gives a contradiction.

(ii) Assume by contradiction that $\alpha \in \mathcal{N}_\Box^*(w_\sim)$, $\alpha \subseteq \beta$ and $\beta \notin \mathcal{N}_\Diamond^*(w_\sim)$. Then $\alpha = [A]_\mathcal{M}^\sim$ for a $A \in \mathcal{L}$ such that $\Box A \in \Phi$ and $[A]_\mathcal{M} \in \mathcal{N}_\Box(w)$. Moreover $\beta = \mathcal{W}^* \setminus [B]_\mathcal{M}^\sim$ for a $B \in \mathcal{L}$ such that $\Diamond B \in \Phi$ and $\mathcal{W} \setminus [B]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$. Thus $[A]_\mathcal{M}^\sim \subseteq \mathcal{W}^* \setminus [B]_\mathcal{M}^\sim$, which implies $[A]_\mathcal{M} \subseteq \mathcal{W} \setminus [B]_\mathcal{M}$. Then, since $\mathcal{M}$ is *strInt*, $\mathcal{W} \setminus [B]_\mathcal{M} \in \mathcal{N}_\Diamond(w)$, which gives a contradiction.

(iii) Assume by contradiction that $-\alpha \in \mathcal{N}_\Box^*(w_\sim)$ and $\mathcal{W}^* \setminus \alpha \notin \mathcal{N}_\Diamond^*(w_\sim)$. Then there is $\Box A \in \Phi$ s.t. $-\alpha = [A]_{\mathcal{M}^*}^\sim$ and $[A]_\mathcal{M} \in \mathcal{N}_\Box(w)$. In addition there is $\Diamond B \in \Phi$ s.t. $\mathcal{W}^* \setminus \alpha = \mathcal{W}^* \setminus [B]_{\mathcal{M}^*}^\sim$ and $\mathcal{W} \setminus [B]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$. As a consequence we have $[A]_{\mathcal{M}^*}^\sim = -[B]_{\mathcal{M}^*}^\sim = [\neg B]_{\mathcal{M}^*}^\sim$. Having $\neg B \in \Phi$, by the filtration lemma we obtain $[A]_\mathcal{M} = [\neg B]_\mathcal{M}$. Then $[\neg B]_\mathcal{M} = -[B]_\mathcal{M} \in \mathcal{N}_\Box(w)$. Finally, by $negInt_a$ $\mathcal{W} \setminus [B]_\mathcal{M} \in \mathcal{N}_\Diamond(w)$, which gives a contradiction.

(iv) Assume by contradiction that $\alpha \in \mathcal{N}_\Box^*(w_\sim)$ and $\mathcal{W}^* \setminus -\alpha \notin \mathcal{N}_\Diamond^*(w_\sim)$. Then there is $\Box A \in \Phi$ s.t. $\alpha = [A]_{\mathcal{M}^*}^\sim$ and $[A]_\mathcal{M} \in \mathcal{N}_\Box(w)$. In addition there is $\Diamond B \in \Phi$ s.t. $\mathcal{W}^* \setminus -\alpha = \mathcal{W}^* \setminus [B]_{\mathcal{M}^*}^\sim$ and $\mathcal{W} \setminus [B]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$. As a consequence we have $[B]_{\mathcal{M}^*}^\sim = -[A]_{\mathcal{M}^*}^\sim = [\neg A]_{\mathcal{M}^*}^\sim$. Having $\neg A \in \Phi$, by the filtration lemma we obtain $[B]_\mathcal{M} = [\neg A]_\mathcal{M}$. Since $\mathcal{M}$ is $negInt_b$, we have $\mathcal{W} \setminus -[A]_\mathcal{M} = \mathcal{W} \setminus [\neg A]_\mathcal{M} = \mathcal{W} \setminus [B]_\mathcal{M} \in \mathcal{N}_\Diamond(w)$, which gives a contradiction.

Lemma 7 *Let $\mathcal{M}$, $\mathcal{M}^*$ and $\mathcal{M}^\circ$ be CINMs, where $\mathcal{M}^*$ is a finest Φ-filtration of $\mathcal{M}$ for a set Φ of formulas that is closed under subformulas. We have:*

(i) If $\mathcal{M}$ is supplemented and weakInt, and $\mathcal{M}^\circ$ is the supplementation of $\mathcal{M}^*$, then $\mathcal{M}^\circ$ is weakInt and is a Φ-filtration of $\mathcal{M}$.
(ii) If $\mathcal{M}$ is closed under intersection and weakInt, and $\mathcal{M}^\circ$ is the intersection closure of $\mathcal{M}^*$, then $\mathcal{M}^\circ$ is weakInt and is a Φ-filtration of $\mathcal{M}$.
(iii) If $\mathcal{M}$ is supplemented, closed under intersection, and weakInt, and $\mathcal{M}^\circ$ is the quasi-filtering of $\mathcal{M}^*$, then $\mathcal{M}^\circ$ is weakInt and is a Φ-filtration of $\mathcal{M}$.
(iv) If $\mathcal{M}$ is closed under intersection and strInt, and $\mathcal{M}^\circ$ is the intersection closure of $\mathcal{M}^*$, then $\mathcal{M}^\circ$ is strInt and is a Φ-filtration of $\mathcal{M}$.
(v) If $\mathcal{M}$ is closed under intersection and $negInt_a$, and $\mathcal{M}^\circ$ is the intersection closure of $\mathcal{M}^*$, and Φ is such that $\neg A \in \Phi$ for all $\Diamond A \in \Phi$, then $\mathcal{M}^\circ$ is $negInt_a$ and is a Φ-filtration of $\mathcal{M}$.

Proof The proofs of (i)–(iv) are very similar to each other. We prove (iii). Firstly we show by contradiction that $\mathcal{M}^\circ$ is *weakInt*. Assume $\alpha \in \mathcal{N}^\circ_\Box(w_\sim)$ and $\alpha \notin \mathcal{N}^\circ_\Diamond(w_\sim)$. Then there are $\alpha_1, ..., \alpha_n \in \mathcal{N}^*_\Box(w_\sim)$ s.t. $\alpha_1 \cap ... \cap \alpha_n \subseteq \alpha$; and there is $\beta \notin \mathcal{N}^*_\Diamond(w_\sim)$ s.t. $\alpha \subseteq \beta$. By definition, this means that there are $\Box A_1, ..., \Box A_n \in \Phi$ s.t. $\alpha_1 = [A_1]^\sim_\mathcal{M}$, ..., $\alpha_n = [A_n]^\sim_\mathcal{M}$, and $[A_1]_\mathcal{M}, ..., [A_n]_\mathcal{M} \in \mathcal{N}_\Box(w)$. Moreover, there is $\Diamond B \in \Phi$ s.t. $\beta = \mathcal{W}^* \setminus [B]^\sim_\mathcal{M}$ and $\mathcal{W} \setminus [B]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$. As a consequence, we also have $[A_1]^\sim_\mathcal{M} \cap ... \cap [A_n]^\sim_\mathcal{M} \subseteq \mathcal{W}^* \setminus [B]^\sim_\mathcal{M}$. Since $\mathcal{M}^*$ is a Φ-filtration of $\mathcal{M}$, by the filtration lemma this implies $[A_1]_\mathcal{M} \cap ... \cap [A_n]_\mathcal{M} \subseteq \mathcal{W} \setminus [B]_\mathcal{M}$. Then by intersection closure of $\mathcal{N}_\Box$, $[A_1]_\mathcal{M} \cap ... \cap [A_n]_\mathcal{M} \in \mathcal{N}_\Box(w)$, and by its supplementation, $\mathcal{W} \setminus [B]_\mathcal{M} \in \mathcal{N}_\Box(w)$. Finally, since $\mathcal{M}$ is *weakInt*, $\mathcal{W} \setminus [B]_\mathcal{M} \in \mathcal{N}_\Diamond(w)$, which gives a contradiction.

We now prove that $\mathcal{M}^\circ$ is a Φ-filtration of $\mathcal{M}$. Let $\Box A \in \Phi$. If $[A]_\mathcal{M} \in \mathcal{N}_\Box(w)$, then $[A]^\sim_\mathcal{M} \in \mathcal{N}^*_\Box(w_\sim)$, and also $[A]^\sim_\mathcal{M} \in \mathcal{N}^\circ_\Box(w_\sim)$. Now assume that $[A]^\sim_\mathcal{M} \in \mathcal{N}^\circ_\Box(w_\sim)$. Then there are $\alpha_1, ..., \alpha_n \in \mathcal{N}^*_\Box(w_\sim)$ s.t. $\alpha_1 \cap ... \cap \alpha_n \subseteq [A]^\sim_\mathcal{M}$. By definition, this means that there are $\Box A_1, ..., \Box A_n \in \Phi$ s.t. $\alpha_1 = [A_1]^\sim_\mathcal{M}$, ..., $\alpha_n = [A_n]^\sim_\mathcal{M}$, and $[A_1]_\mathcal{M}, ..., [A_n]_\mathcal{M} \in \mathcal{N}_\Box(w)$. Then, since $\mathcal{M}^*$ is a Φ-filtration of $\mathcal{M}$, $[A_1]_\mathcal{M} \cap ... \cap [A_n]_\mathcal{M} \subseteq [A]_\mathcal{M}$. By intersection closure of $\mathcal{N}_\Box$, $[A_1]_\mathcal{M} \cap ... \cap [A_n]_\mathcal{M} \in \mathcal{N}_\Box(w)$, then by supplementation, $[A]_\mathcal{M} \in \mathcal{N}_\Box(w)$.

Now let $\Diamond A \in \Phi$. If $\mathcal{W} \setminus [A]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$, then $\mathcal{W}^* \setminus [A]^\sim_\mathcal{M} \notin \mathcal{N}^*_\Diamond(w_\sim)$, and also $\mathcal{W}^* \setminus [A]^\sim_\mathcal{M} \notin \mathcal{N}^\circ_\Diamond(w_\sim)$. Now assume $\mathcal{W}^* \setminus [A]^\sim_\mathcal{M} \notin \mathcal{N}^\circ_\Diamond(w_\sim)$. Then there is $\beta \notin \mathcal{N}^*_\Diamond(w_\sim)$ s.t. $\mathcal{W}^* \setminus [A]^\sim_\mathcal{M} \subseteq \beta$. By definition, $\beta = \mathcal{W}^* \setminus [B]^\sim_\mathcal{M}$ for a $\Diamond B \in \Phi$ s.t. $\mathcal{W} \setminus [B]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$. Since $\mathcal{M}^*$ is a Φ-filtration of $\mathcal{M}$, we have $\mathcal{W} \setminus [A]_\mathcal{M} \subseteq \mathcal{W} \setminus [B]_\mathcal{M}$. Then by supplementation, $\mathcal{W} \setminus [A]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$.

(v) Assume by contradiction that $-\alpha \in \mathcal{N}^\circ_\Box(w_\sim)$ and $\mathcal{W}^* \setminus \alpha \notin \mathcal{N}^\circ_\Diamond(w_\sim)$. Then there are $\alpha_1, ..., \alpha_n \in \mathcal{N}^*_\Box(w_\sim)$ s.t. $\alpha_1 \cap ... \cap \alpha_n = -\alpha$; in addition $\mathcal{W}^* \setminus \alpha \notin \mathcal{N}^*_\Diamond(w_\sim)$. By definition there are $\Box A_1, ..., \Box A_n, \Diamond B \in \Phi$ s.t. $\alpha_1 = [A_1]^\sim_\mathcal{M}$, ..., $\alpha_n = [A_n]^\sim_\mathcal{M}$, and $[A_1]_\mathcal{M}, ..., [A_n]_\mathcal{M} \in \mathcal{N}_\Box(w)$; moreover $\mathcal{W}^* \setminus \alpha = \mathcal{W}^* \setminus [B]^\sim_\mathcal{M}$ and $\mathcal{W} \setminus [B]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$. Thus $[A_1]^\sim_\mathcal{M} \cap ... \cap [A_n]^\sim_\mathcal{M} = -[B]^\sim_\mathcal{M} = [\neg B]^\sim_\mathcal{M}$. Since $\mathcal{M}^*$ is a Φ-filtration of $\mathcal{M}$ and $\neg B \in \Phi$, by the filtration lemma this implies $[A_1]_\mathcal{M} \cap ... \cap [A_n]_\mathcal{M} = [\neg B]_\mathcal{M} = -[B]_\mathcal{M}$. But by intersection closure of $\mathcal{N}_\Box$, $[A_1]_\mathcal{M} \cap ... \cap [A_n]_\mathcal{M} \in \mathcal{N}_\Box(w)$, then by $negInt_a$, $\mathcal{W} \setminus [B]_\mathcal{M} \in \mathcal{N}_\Diamond(w)$, which gives a contradiction. Similarly to (iii) we can also prove that $\mathcal{M}^\circ$ is a Φ-filtration of $\mathcal{M}$.

Theorem 13 *If a formula A is satisfiable in a CINM $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$, then A is satisfiable in a CINM $\mathcal{M}' = \langle \mathcal{W}', \preceq', \mathcal{N}'_\Box, \mathcal{N}'_\Diamond, \mathcal{V}' \rangle$, where $\mathcal{N}'_\Box$ and $\mathcal{N}'_\Diamond$ have the same properties of $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$, and $\mathcal{W}'$ is finite.*

Proof The proof is standard, by taking $\Phi = Sbf(A) \cup \Psi$, where $Sbf(A)$ is the set of subformulas of A, and Ψ depends on the properties of $\mathcal{M}$. In particular, Ψ contains $\Diamond\bot, \bot$ if $\mathcal{N}_\Diamond$ contains the unit; it contains $\Box\top, \top, \bot$ if $\mathcal{N}_\Box$ contains the unit; it contains $\neg B$ for all $\Diamond B \in Sbf(A)$ if $\mathcal{M}$ is $negInt_a$ (and not $strInt$); and it contains $\neg B$ for all $\Box B \in Sbf(A)$ if $\mathcal{M}$ is $negInt_b$ (and not $strInt$). Moreover, depending on the properties of $\mathcal{M}$ we consider the right transformation $\mathcal{M}'$ of $\mathcal{M}$. Observe that the set Φ is always finite, which implies that any Φ-filtration $\mathcal{M}'$ of $\mathcal{M}$ is a finite model.

Corollary 1 *Any intuitionistic non-normal mono- or bi-modal logic enjoys the finite model property.*

7 Constructive K and propositional CCDL

We have seen in Section 6 that $\Box$-INMs coincide essentially with Goldblatt's neighbourhood spaces. In Fairtlough and Mendler [9], Goldblatt's spaces are considered in order to provide a semantics for Propositional Lax Logic (PLL). PLL is an intuitionistic monomodal logic which is motivated by hardware verification, and is non-normal as it fails to validate the rule of necessitation.

In this section we show that the framework of CINMs is *general enough* to cover two additional intuitionistic non-normal bimodal logics, namely CK (for "constructive K") by Bellin *et al.* [4], and the propositional fragment of Wijesekera's first-order logic CCDL (Wijesekera [42]), that we call $\mathsf{CCDL^p}$. In particular, we show that the two systems can be captured in our framework by considering a very simple additional property.

Different possible worlds semantics have already been proposed for the two logics. In particular, the logic $\mathsf{CCDL^p}$ has both a relational semantics (Wijesekera [42]) and a neighbourhood semantics (Kojima [24]), whereas a relational semantics for CK has been given in Mendler and de Paiva [33] by adding *inconsistent* worlds to the relational models for $\mathsf{CCDL^p}$. The fact that CK and $\mathsf{CCDL^p}$ fit in our framework is interesting for two reasons. On the one hand, it shows the power of our neighbourhood semantics, that can accommodate in a natural way many systems. On the other hand, it shows that CK and $\mathsf{CCDL^p}$ can be obtained as extensions of weaker logics in a modular way. As a further advantage, observe that this semantics is standard also for CK, as our models do not need inconsistent worlds.

In the following, we first present the logics CK and $\mathsf{CCDL^p}$ by giving both their axiomatisations and sequent calculi. Then we define their CINMs and prove soundness and completeness, as well as the finite model property. Finally, we present their pre-existing possible worlds semantics and prove directly their equivalence with CINMs.

7.1 Hilbert systems and sequent calculi

Logic CK (Bellin *et al.* [4]) is defined as a Hilbert system by adding to IPL the following axioms and rules:

$$\mathsf{K}_\Box \quad \Box(A \supset B) \supset (\Box A \supset \Box B) \qquad \mathsf{K}_\Diamond \quad \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) \qquad \mathsf{Nec}\ \frac{A}{\Box A}\ .$$

The logic $\mathsf{CCDL^p}$ is the extension of CK with axiom $\mathsf{N}_\Diamond$ $(\neg\Diamond\bot)$.[4] It is worth noticing that, given the syntactical equivalences that we have recalled in Section 3, an equivalent axiomatisation for CK is obtained by extending IPL with the rules $\mathsf{E}_\Box$ and $\mathsf{E}_\Diamond$, and axioms $\mathsf{M}_\Box$, $\mathsf{N}_\Box$, $\mathsf{C}_\Box$, and $\mathsf{K}_\Diamond$. As before, by adding also $\mathsf{N}_\Diamond$ we obtain the logic $\mathsf{CCDL^p}$. Notice that the axiom $\mathsf{M}_\Diamond$ is derivable in both systems, *e.g.* from Nec and $\mathsf{K}_\Diamond$.

Both logics CK and $\mathsf{CCDL^p}$ are non-normal, as they reject some form of distributivity of $\Diamond$ over $\vee$. In particular, $\mathsf{CCDL^p}$ rejects binary distributivity ($\mathsf{C}_\Diamond$), while CK rejects both binary and nullary distributivity ($\mathsf{C}_\Diamond$, $\mathsf{N}_\Diamond$). By contrast, the modality $\Box$ is normal, as the systems contain the axiom $\mathsf{K}_\Box$ and the rule of necessitation.

The sequent calculi for CK and $\mathsf{CCDL^p}$ (denoted here as G.CK and $\mathsf{G.CCDL^p}$) are defined, respectively, in Bellin *et al.* [4] and in Wijesekera [42]. In order to present the calculi, we consider the following rule, that we call $\mathsf{W^{seq}}$ (for "Wijesekera"):

$$\mathsf{W^{seq}}\ \frac{A_1, ..., A_n, B \Rightarrow C}{\Gamma, \Box A_1, ..., \Box A_n, \Diamond B \Rightarrow \Diamond C} \quad (n \geq 1)\ .$$

Both [4] and [42] allow the set $\{A_1, ..., A_n\}$ in $\mathsf{W^{seq}}$ to be empty, thus including implicitly $\mathsf{M}_\Diamond^{\mathsf{seq}}$. By uniformity with the formulation of the other rules, we require it to contain at least one formula. Then, given the present formulation, G.CK and $\mathsf{G.CCDL^p}$ are defined by extending G3ip as follows:

$$\begin{array}{lll} \mathsf{G.CK} & := & \mathsf{M}_\Box\mathsf{C^{seq}} + \mathsf{M}_\Diamond^{\mathsf{seq}} + \mathsf{N}_\Box^{\mathsf{seq}} + \mathsf{W^{seq}} \\ \mathsf{G.CCDL^p} & := & \mathsf{M}_\Box\mathsf{C^{seq}} + \mathsf{M}_\Diamond^{\mathsf{seq}} + \mathsf{N}_\Box^{\mathsf{seq}} + \mathsf{W^{seq}} + \mathsf{strC^{seq}} + \mathsf{N}_\Diamond^{\mathsf{seq}} \end{array}$$

Observe that $\mathsf{G.CCDL^p}$ can be seen as an extension of our top calculus $\mathsf{G.IMCN}_\Box$, as it corresponds to $\mathsf{G.IMCN}_\Box + \mathsf{W^{seq}}$. Instead, G.CK is not comparable with any of our bimodal calculi, as it contains the rule $\mathsf{N}_\Box^{\mathsf{seq}}$ but it does not contain $\mathsf{N}_\Diamond^{\mathsf{seq}}$, a case which never occurs in any calculus of our lattice.

Theorem 14 ([4] for G.CK, [42] for $\mathsf{G.CCDL^p}$) *The rule* cut *is admissible in G.CK and $\mathsf{G.CCDL^p}$. Moreover, G.CK and $\mathsf{G.CCDL^p}$ are equivalent with their corresponding axiomatisations.*

Notice that having $\mathsf{W^{seq}}$ instead of our "weak interaction" rules allows us to take $\mathsf{N}_\Box^{\mathsf{seq}}$ and not $\mathsf{N}_\Diamond^{\mathsf{seq}}$ (as in G.CK), and to still obtain a cut-free calculus. If instead we take both $\mathsf{W^{seq}}$ and $\mathsf{N}_\Diamond^{\mathsf{seq}}$ (as in $\mathsf{G.CCDL^p}$), we need to consider also $\mathsf{strC^{seq}}$ in order to have the admissibility of the cut rule, as explicited by the following derivation:

[4] The axiomatisation given by Wijesekera [42] includes also $\Diamond(A \supset B) \supset (\Box A \supset \Diamond B)$; however this formula is derivable from the other axioms (cf. *e.g.* Simpson [38], p. 48).

$$\dfrac{\mathsf{W^{seq}}\ \dfrac{p, \neg p \Rightarrow \bot}{\Box p, \Diamond\neg p \Rightarrow \Diamond\bot} \qquad \dfrac{\bot \Rightarrow}{\Diamond\bot \Rightarrow}\ \mathsf{N^{seq}_{\Diamond}}}{\Box p, \Diamond\neg p \Rightarrow}\ \mathsf{cut}$$

It is easy to verify that the endsequent $\Box p, \Diamond\neg p \Rightarrow$ is derivable in $\mathsf{G.CCDL^p} \setminus \{\mathsf{strC^{seq}}\}$ if and only if the cut rule is applied, but it has a cut-free derivation in $\mathsf{G.CCDL^p}$ by applying $\mathsf{strC^{seq}}$ to $p, \neg p \Rightarrow$. Notice also that adding $\mathsf{strC^{seq}}$ to the calculus preserve the equivalence with the axiomatisation, since str is derivable from $\mathsf{K}_\Diamond$, $\mathsf{Mon}_\Diamond$ and $\mathsf{N}_\Diamond$.

7.2 Intuitionistic neighbourhood models for CK and $\mathsf{CCDL^p}$

We now define CINMs for CK and $\mathsf{CCDL^p}$, and prove the soundness and completeness of both systems.

Definition 10 (Intuitionistic neighbourhood models for CK and $\mathsf{CCDL^p}$) A CINM for CK (CK-model in the following) is any CINM in which $\mathcal{N}_\Box$ is supplemented, closed under intersection and contains the unit; $\mathcal{N}_\Diamond$ is supplemented; and such that:

$$\text{If } \alpha \in \mathcal{N}_\Box(w) \text{ and } \beta \in \mathcal{N}_\Diamond(w), \text{ then } \alpha \cap \beta \in \mathcal{N}_\Diamond(w) \qquad (WInt).$$

A CINM for $\mathsf{CCDL^p}$ ($\mathsf{CCDL^p}$-model in the following) is any CINM for CK satisfying also the condition of *weakInt* ($\mathcal{N}_\Box(w) \subseteq \mathcal{N}_\Diamond(w)$).

Notice that, as a consequence, the function $\mathcal{N}_\Diamond$ in $\mathsf{CCDL^p}$-models contains the unit. We now prove that the logics CK and $\mathsf{CCDL^p}$ are sound and complete with respect to the corresponding models.

Theorem 15 (Soundness) *The logics CK and $\mathsf{CCDL^p}$ are sound with respect to CK- and $\mathsf{CCDL^p}$-models, respectively.*

Proof We just consider the axiom $\mathsf{K}_\Diamond$. Assume that $w \Vdash \Box(A \supset B)$ and that $w \not\Vdash \Diamond B$. Then $[A \supset B] \in \mathcal{N}_\Box(w)$ and $\mathcal{W} \setminus [B] \in \mathcal{N}_\Diamond(w)$. By $WInt$, $[A \supset B] \cap (\mathcal{W} \setminus [B]) \in \mathcal{N}_\Diamond(w)$. Since $[A \supset B] \cap (\mathcal{W} \setminus [B]) \subseteq (\mathcal{W} \setminus [A])$, by supplementation we have $\mathcal{W} \setminus [A] \in \mathcal{N}_\Diamond(w)$; therefore $w \not\Vdash \Diamond A$.

Completeness is proved as before by the canonical model construction.

Lemma 8 *Let the canonical models $\mathcal{M}^c_{\mathsf{CK}}$ for CK, and $\mathcal{M}^c_{\mathsf{CCDL^p}}$ for $\mathsf{CCDL^p}$, be defined as in Definition 7. Then $\mathcal{M}^c_{\mathsf{CK}}$ and $\mathcal{M}^c_{\mathsf{CCDL^p}}$ are, respectively, a CK-model and a $\mathsf{CCDL^p}$-model.*

Proof We show that both $\mathcal{M}^c_{\mathsf{CK}}$ and $\mathcal{M}^c_{\mathsf{CCDL^p}}$ satisfy the condition of $WInt$. Assume $\alpha \in \mathcal{N}^+_\Box(X)$ and $\alpha \cap \beta \notin \mathcal{N}^+_\Diamond(X)$. Then there are $A, B \in \mathcal{L}$ such that $\uparrow_{pr} A \subseteq \alpha$, $\alpha \cap \beta \subseteq \mathcal{W}^c \setminus \uparrow_{pr} B$ and $\Box A, \Diamond B \in X$. As a consequence, $\uparrow_{pr} A \cap \beta \subseteq \mathcal{W}^c \setminus \uparrow_{pr} B$, that by standard properties of set inclusion implies $\beta \subseteq (\mathcal{W}^c \setminus \uparrow_{pr} A) \cup (\mathcal{W}^c \setminus \uparrow_{pr} B) = \mathcal{W}^c \setminus \uparrow_{pr} (A \wedge B)$. Moreover, since $(\Box A \wedge \Diamond B) \supset \Diamond(A \wedge B)$ is derivable (from $A \supset (B \supset A \wedge B)$, by $\mathsf{Mon}_\Box$ and $\mathsf{K}_\Diamond$), we have $\Diamond(A \wedge B) \in X$. Thus, by definition, $\beta \notin \mathcal{N}^+_\Diamond(X)$. In addition, by Lemma 3 (iv), $\mathcal{M}^c_{\mathsf{CCDL^p}}$ is also *weakInt*, as str is derivable in $\mathsf{CCDL^p}$.

Theorem 16 (Completeness) *Logics CK and CCDL^p are complete with respect to CK- and CCDL^p-models, respectively.*

Proof Same proof as for Theorem 11, using this time Lemma 8.

By applying the filtration technique to CK- and CCDL^p-models we also prove that both systems enjoy the finite model property. For CK, FMP with respect to the original relational semantics has been proved in Mendler and de Paiva [33], whereas — to the best of our knowledge — an analogous result has not been stated explicitly before for CCDL^p.

Lemma 9 *Let $\mathcal{M}$ and $\mathcal{M}^*$ be CINMs, where $\mathcal{M}^*$ is a finest Φ-filtration of $\mathcal{M}$ for a set Φ of formulas that is closed under subformulas and contains $\Box\top, \Diamond\bot$. We call $WInt$* closure *of $\mathcal{M}^*$ any CINM $\mathcal{M}^\circ = \langle \mathcal{W}^*, \preceq^*, \mathcal{N}_\Box^\circ, \mathcal{N}_\Diamond^\circ, \mathcal{V}^* \rangle$ such that*

$\alpha \in \mathcal{N}_\Box^\circ(w_\sim)$ *iff* *there are* $\alpha_1, ..., \alpha_n \in \mathcal{N}_\Box^*(w_\sim)$ *s.t.* $\alpha_1 \cap ... \cap \alpha_n \subseteq \alpha$;
$\alpha \notin \mathcal{N}_\Diamond^\circ(w_\sim)$ *iff* *there are* $\beta_1, ..., \beta_n \in \mathcal{N}_\Box^*(w_\sim)$ *and* $\gamma \notin \mathcal{N}_\Diamond^*(w_\sim)$ *s.t.* $\alpha \cap \beta_1 \cap ... \cap \beta_n \subseteq \gamma$.

The following hold:
(i) If $\mathcal{M}$ is a CK-model, then $\mathcal{M}^\circ$ is a CK-model.
(ii) If $\mathcal{M}$ is a CCDL^p-model, then $\mathcal{M}^\circ$ is a CCDL^p-model.
(iii) If $\mathcal{M}$ is a CK- or a CCDL^p-model, then $\mathcal{M}^\circ$ is a Φ-filtration of $\mathcal{M}$.

Proof (i) Clearly $\mathcal{N}_\Box^\circ$ is supplemented and closed under intersection, and it is immediate to check that $\mathcal{N}_\Diamond^\circ$ is supplemented. By Lemma 5 we also have that $\mathcal{N}_\Box^\circ$ contains the unit. Here we show that $\mathcal{M}^\circ$ satisfies $WInt$. In this respect, we assume that $\alpha \in \mathcal{N}_\Box^\circ(w_\sim)$ and that $\alpha \cap \beta \notin \mathcal{N}_\Diamond^\circ(w_\sim)$. By definition, there are $\alpha_1, ..., \alpha_n \in \mathcal{N}_\Box^*(w_\sim)$ s.t. $\alpha_1 \cap ... \cap \alpha_n \subseteq \alpha$. Moreover, there are $\beta_1, ..., \beta_k \in \mathcal{N}_\Box^*(w_\sim)$ and $\gamma \notin \mathcal{N}_\Diamond^*(w_\sim)$ s.t. $(\alpha \cap \beta) \cap \beta_1 \cap ... \cap \beta_k \subseteq \gamma$. This implies that $\alpha_1 \cap ... \cap \alpha_n \cap \beta \cap \beta_1 \cap ... \cap \beta_k \subseteq \gamma$. Therefore $\beta \notin \mathcal{N}_\Diamond^\circ(w_\sim)$.

(ii) In addition to the properties of (i), we prove here that $\mathcal{M}^\circ$ is also *weakInt*. Assume by contradiction that $\alpha \in \mathcal{N}_\Box^\circ(w_\sim)$ and $\alpha \notin \mathcal{N}_\Diamond^\circ(w_\sim)$. Then there are $\alpha_1, ..., \alpha_n \in \mathcal{N}_\Box^*(w_\sim)$ s.t. $\alpha_1 \cap ... \cap \alpha_n \subseteq \alpha$. Moreover, there are $\beta_1, ..., \beta_n \in \mathcal{N}_\Box^*(w_\sim)$ and $\gamma \notin \mathcal{N}_\Diamond^*(w_\sim)$ s.t. $\alpha \cap \beta_1 \cap ... \cap \beta_n \subseteq \gamma$. This implies that there are $\Box A_1, ..., \Box A_n, \Box B_1, ..., \Box B_k, \Diamond C \in \Phi$ s.t. $\alpha_1 = [A_1]^\sim_{\mathcal{M}^*}, ..., \alpha_n = [A_n]^\sim_{\mathcal{M}^*}, \beta_1 = [B_1]^\sim_{\mathcal{M}^*}, ..., \beta_k = [B_k]^\sim_{\mathcal{M}^*}$, and $\gamma = \mathcal{W}^* \setminus [C]^\sim_{\mathcal{M}^*}$. In addition, $[A_1]_\mathcal{M}, ..., [A_n]_\mathcal{M}, [B_1]_\mathcal{M}, ..., [B_k]_\mathcal{M} \in \mathcal{N}_\Box(w)$ and $\mathcal{W} \setminus [C]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$. By the filtration lemma, we obtain $[A_1]_\mathcal{M} \cap ... \cap [A_n]_\mathcal{M} \cap [B_1]_\mathcal{M} \cap ... \cap [B_k]_\mathcal{M} \subseteq \mathcal{W} \setminus [C]_\mathcal{M}$. Finally, since $\mathcal{N}_\Box$ is supplemented and closed under intersection, and $\mathcal{M}$ is *weakInt*, we have $\mathcal{W} \setminus [C]_\mathcal{M} \in \mathcal{N}_\Diamond(w)$, which leads to a contradiction.

(iii) For $\Box A \in \Phi$, the proof is exactly as in Lemma 7. Let $\Diamond A \in \Phi$. If $\mathcal{W} \setminus [A]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$, then $\mathcal{W}^* \setminus [A]^\sim_{\mathcal{M}^*} \notin \mathcal{N}_\Diamond^*(w_\sim)$. Thus, since by Lemma 5 $\mathcal{W}^* \in \mathcal{N}_\Box(w_\sim)$, we have $\mathcal{W}^* \setminus [A]^\sim_{\mathcal{M}^*} \notin \mathcal{N}_\Diamond^\circ(w_\sim)$. Now assume that $\mathcal{W}^* \setminus [A]^\sim_{\mathcal{M}^*} \notin \mathcal{N}_\Diamond^\circ(w_\sim)$. Then there are $\beta_1, ..., \beta_n \in \mathcal{N}_\Box^*(w_\sim)$ and $\gamma \notin \mathcal{N}_\Diamond^*(w_\sim)$ s.t. $\mathcal{W}^* \setminus [A]^\sim_{\mathcal{M}^*} \cap \beta_1 \cap ... \cap \beta_n \subseteq \gamma$. Hence, by definition, there exist $\Box A_1, ..., \Box A_n, \Box B_1, ..., \Box B_k, \Diamond C \in \Phi$ s.t. $\beta_1 = [B_1]^\sim_{\mathcal{M}^*}, ..., \beta_k = [B_k]^\sim_{\mathcal{M}^*}$, and $\gamma = \mathcal{W}^* \setminus [C]^\sim_{\mathcal{M}^*}$. In addition, $[B_1]_\mathcal{M}, ..., [B_k]_\mathcal{M} \in \mathcal{N}_\Box(w)$ and $\mathcal{W} \setminus [C]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$. By contradiction, assume that $\mathcal{W} \setminus [A]_\mathcal{M} \in \mathcal{N}_\Diamond(w)$. Then, by intersection closure of $\mathcal{N}_\Box$ and $WInt$, $[B_1]_\mathcal{M} \cap ... \cap [B_k]_\mathcal{M} \cap \mathcal{W} \setminus [A]_\mathcal{M} \in \mathcal{N}_\Diamond(w)$. Moreover, by the filtration lemma, we have that $[B_1]_\mathcal{M} \cap ... \cap [B_k]_\mathcal{M} \cap \mathcal{W} \setminus [A]_\mathcal{M} \subseteq \mathcal{W} \setminus [C]_\mathcal{M}$. Thus, by supplementation of $\mathcal{N}_\Diamond$, we obtain that $\mathcal{W} \setminus [C]_\mathcal{M} \in \mathcal{N}_\Diamond(w)$, which leads to a contradiction.

We can then prove in a standard way the following theorem:

Theorem 17 *CK and CCDL$^\mathsf{p}$ enjoy the finite model property.*

7.3 Pre-existing semantics and direct proofs of equivalence

7.3.1 Semantic equivalence for CCDL$^\mathsf{p}$

The results of the previous section show that CK and CCDL$^\mathsf{p}$ are equally characterised by our neighbourhood semantics and by the original ones, given respectively by Mendler and de Paiva [33] and Wijesekera [42]. It is instructive, however, to prove the equivalence directly by mutual transformations of models. We begin with system CCDL$^\mathsf{p}$, and consider the relational models by Wijesekera [42] as well as the neighbourhood models by Kojima [24].

Definition 11 (Relational models for CCDL$^\mathsf{p}$ (Wijesekera [42])) A relational model for CCDL$^\mathsf{p}$ is a tuple $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{R}, \mathcal{V} \rangle$, where $\mathcal{W}$, $\preceq$ and $\mathcal{V}$ are as in Definition 5, and $\mathcal{R}$ is any binary relation on $\mathcal{W}$. The forcing relation $w \Vdash_r A$ is defined as $w \Vdash A$ (Definition 5) for $A \equiv p, B \wedge C, B \vee C, B \supset C$; and in the following way for modal formulas:

$w \Vdash_r \Box B$ iff for all $v \succeq w$, for all $u \in \mathcal{W}$, $v\mathcal{R}u$ implies $u \Vdash_r B$;
$w \Vdash_r \Diamond B$ iff for all $v \succeq w$, there is $u \in \mathcal{W}$ s.t. $v\mathcal{R}u$ and $u \Vdash_r B$.

Definition 12 (Kojima's neighbourhood models for CCDL$^\mathsf{p}$ (Kojima [24])) Kojima's neighbourhood models for CCDL$^\mathsf{p}$ are tuples $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_k, \mathcal{V} \rangle$, where $\mathcal{W}$, $\preceq$ and $\mathcal{V}$ are as in Definition 5, and $\mathcal{N}_k$ is a neighbourhood function $\mathcal{W} \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}))$ such that:

- $w \preceq v$ implies that $\mathcal{N}_k(v) \subseteq \mathcal{N}_k(w)$;
- $\mathcal{N}_k(w) \neq \emptyset$ for all $w \in \mathcal{W}$.

The forcing relation $w \Vdash_k A$ is defined as usual for $A \equiv p, \bot, B \wedge C, B \vee C, B \supset C$; and for modal formulas it is defined as follows:

$w \Vdash_k \Box B$ iff for all $\alpha \in \mathcal{N}_k(w)$, for all $v \in \alpha$, $v \Vdash_k B$;
$w \Vdash_k \Diamond B$ iff for all $\alpha \in \mathcal{N}_k(w)$, there is $v \in \alpha$ s.t. $v \Vdash_k B$.

Theorem 18 (Wijesekera [42], Kojima [24]) *The logic CCDL$^\mathsf{p}$ is sound and complete with respect to relational models for CCDL$^\mathsf{p}$, as well as with respect to Kojima's models for CCDL$^\mathsf{p}$.*

That relational models, Kojima's models and CINMs for $\mathsf{CCDL^p}$ are equivalent is a corollary of the respective completeness theorems. It is instructive, however, to prove the equivalence directly. A proof of equivalence of Kojima's and relational models is given in Kojima [24]. Here we prove directly the equivalence of Kojima's and CINMs for $\mathsf{CCDL^p}$. In particular, we show that every Kojima model can be transformed into an equivalent CINM for $\mathsf{CCDL^p}$, and that every *finite* CINM for $\mathsf{CCDL^p}$ can be transformed into an equivalent Kojima model. By combining these results with the transformations given by Kojima we also obtain direct transformations between CINMs and relational models. Furthermore, considering also the finite model property of $\mathsf{CCDL^p}$ with respect to the corresponding CINMs (cf. Theorem 17), this provides an alternative proof of equivalence of the three semantics.

In the proof of some of the next lemmas we shall make use of the following property, which is satisfied by any finite model for $\mathsf{CCDL^p}$ and CK, and is an easy consequence of $WInt$ and the intersection closure of $\mathcal{N}_\Box$.

Fact 2. Every finite CINM for $\mathsf{CCDL^p}$ or for CK satisfies the following property: For all $\alpha \in \mathcal{N}_\Diamond(w)$, there is $\beta \in \mathcal{N}_\Diamond(w)$ s.t. $\beta \subseteq \alpha$ and $\beta \subseteq \bigcap \mathcal{N}_\Box(w)$ $(WInt')$.

Lemma 10 *Let $\mathcal{M}_k = \langle \mathcal{W}, \preceq, \mathcal{N}_k, \mathcal{V}\rangle$ be a Kojima model for $\mathsf{CCDL^p}$, and let $\mathcal{M}_n$ be the model $\langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V}\rangle$ where $\mathcal{W}$, $\preceq$ and $\mathcal{V}$ are as in $\mathcal{M}_k$, and:*

$\mathcal{N}_\Box(w) = \{\alpha \subseteq \mathcal{W} \mid \bigcup \mathcal{N}_k(w) \subseteq \alpha\}$;
$\mathcal{N}_\Diamond(w) = \{\alpha \subseteq \mathcal{W} \mid \text{there is } \beta \in \mathcal{N}_k(w) \text{ s.t. } \beta \subseteq \alpha\}$.

Then $\mathcal{M}_n$ is a CINM for $\mathsf{CCDL^p}$ and is pointwise equivalent to $\mathcal{M}_k$.

Proof It is immediate to verify that $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ are supplemented and contain the unit; that $\mathcal{N}_\Box$ is closed under intersection; and that $w \preceq v$ implies $\mathcal{N}_\Box(w) \subseteq \mathcal{N}_\Box(v)$ and $\mathcal{N}_\Diamond(v) \subseteq \mathcal{N}_\Diamond(w)$. We show that $\mathcal{M}_n$ satisfies the other properties of $\mathsf{CCDL^p}$-models.

$(weakInt)$ Assume $\alpha \in \mathcal{N}_\Box(w)$. Then $\bigcup \mathcal{N}_k(w) \subseteq \alpha$, and, since $\mathcal{N}_k(w) \neq \emptyset$, there is $\beta \in \mathcal{N}_k(w)$ such that $\beta \subseteq \alpha$. Therefore $\alpha \in \mathcal{N}_\Diamond(w)$.

$(WInt)$ Assume $\alpha \in \mathcal{N}_\Box(w)$ and $\beta \in \mathcal{N}_\Diamond(w)$. Then $\bigcup \mathcal{N}_k(w) \subseteq \alpha$ and there is $\gamma \in \mathcal{N}_k(w)$ such that $\gamma \subseteq \beta$. Thus $\gamma \subseteq \bigcup \mathcal{N}_k(w)$, which implies $\gamma \subseteq \alpha \cap \beta$. Therefore $\alpha \cap \beta \in \mathcal{N}_\Diamond(w)$.

We now prove by induction on A that for every $A \in \mathcal{L}$ and $w \in \mathcal{W}$,

$$\mathcal{M}_n, w \Vdash A \ \text{ iff } \ \mathcal{M}_k, w \Vdash_k A.$$

We only consider the inductive cases $A \equiv \Box B, \Diamond B$.

$A \equiv \Box B$. $\mathcal{M}_n, w \Vdash \Box B$ iff $[B]_{\mathcal{M}_n} \in \mathcal{N}_\Box(w)$ iff $\bigcup \mathcal{N}_k(w) \subseteq [B]_{\mathcal{M}_n}$ iff (i.h.) $\bigcup \mathcal{N}_k(w) \subseteq [B]_{\mathcal{M}_k}$ iff for all $\alpha \in \mathcal{N}_k(w)$, $\alpha \subseteq [B]_{\mathcal{M}_k}$ iff $\mathcal{M}_k, w \Vdash_k \Box B$.

$A \equiv \Diamond B$. $\mathcal{M}_n, w \Vdash \Diamond B$ iff $\mathcal{W} \setminus [B]_{\mathcal{M}_n} \notin \mathcal{N}_\Diamond(w)$ iff for all $\alpha \in \mathcal{N}_k(w)$, $\alpha \cap [B]_{\mathcal{M}_n} \neq \emptyset$ iff (i.h.) for all $\alpha \in \mathcal{N}_k(w)$, $\alpha \cap [B]_{\mathcal{M}_k} \neq \emptyset$ iff $\mathcal{M}_k, w \Vdash_k \Diamond B$.

Lemma 11 *Let $\mathcal{M}_n = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V}\rangle$ be a finite CINM for $\mathsf{CCDL^p}$, and let $\mathcal{M}_k$ be the model $\langle \mathcal{W}, \preceq, \mathcal{N}_k, \mathcal{V}\rangle$ where $\mathcal{W}$, $\preceq$ and $\mathcal{V}$ are as in $\mathcal{M}_n$, and:*

$\mathcal{N}_k(w) = \{\alpha \in \mathcal{N}_\Diamond(w) \mid \alpha \subseteq \bigcap \mathcal{N}_\Box(w)\}$.

Then $\mathcal{M}_k$ is a Kojima model for $\mathsf{CCDL^p}$ and is pointwise equivalent to $\mathcal{M}_n$.

Proof First, notice that $\mathcal{M}_k$ is a Kojima model: by intersection closure, we have that $\bigcap \mathcal{N}_\Box(w) \in \mathcal{N}_\Box(w)$, hence by *weakInt*, $\bigcap \mathcal{N}_\Box(w) \in \mathcal{N}_\Diamond(w)$. Thus $\bigcap \mathcal{N}_\Box(w) \in \mathcal{N}_k(w)$, which implies $\mathcal{N}_k(w) \neq \emptyset$. Moreover, assume that $w \preceq v$ and $\alpha \in \mathcal{N}_k(v)$. It follows that $\alpha \in \mathcal{N}_\Diamond(v)$ and $\alpha \subseteq \bigcap \mathcal{N}_\Box(v)$. Since $\mathcal{N}_\Diamond(v) \subseteq \mathcal{N}_\Diamond(w)$ and $\mathcal{N}_\Box(w) \subseteq \mathcal{N}_\Box(v)$, we have both $\alpha \in \mathcal{N}_\Diamond(w)$ and $\alpha \subseteq \bigcap \mathcal{N}_\Box(w)$, therefore $\alpha \in \mathcal{N}_k(w)$.

We prove by induction on A that for every $A \in \mathcal{L}$ and $w \in \mathcal{W}$,

$$\mathcal{M}_n, w \Vdash A \ \text{ iff } \ \mathcal{M}_k, w \Vdash_k A.$$

As before, we only consider the inductive cases $A \equiv \Box B, \Diamond B$:

$A \equiv \Box B$. $\mathcal{M}_k, w \Vdash_k \Box B$ iff for all $\alpha \in \mathcal{N}_k(w)$, $\alpha \subseteq [B]_{\mathcal{M}_k}$ iff (since $\bigcap \mathcal{N}_\Box(w) \in \mathcal{N}_k(w)$) $\bigcap \mathcal{N}_\Box(w) \subseteq [B]_{\mathcal{M}_k}$ iff (i.h.) $\bigcap \mathcal{N}_\Box(w) \subseteq [B]_{\mathcal{M}_n}$ iff (by properties of $\mathcal{N}_\Box(w)$) $[B]_{\mathcal{M}_n} \in \mathcal{N}_\Box(w)$ iff $\mathcal{M}_n, w \Vdash \Box B$.

$A \equiv \Diamond B$. Assume $\mathcal{M}_k, w, \Vdash_k \Diamond B$. Then for every $\alpha \in \mathcal{N}_k(w)$, $\alpha \cap [B]_{\mathcal{M}_k} \neq \emptyset$, and, by inductive hypothesis, $\alpha \cap [B]_{\mathcal{M}_n} \neq \emptyset$. Thus for every $\alpha \in \mathcal{N}_\Diamond(w)$ s.t. $\alpha \subseteq \bigcap \mathcal{N}_\Box(w)$, $\alpha \cap [B]_{\mathcal{M}_n} \neq \emptyset$. Let β be any neighbourhood in $\mathcal{N}_\Diamond(w)$. By $WInt'$, there is $\gamma \subseteq \beta$ s.t. $\gamma \in \mathcal{N}_\Diamond(w)$ and $\gamma \subseteq \bigcap \mathcal{N}_\Box(w)$. Then $\gamma \cap [B]_{\mathcal{M}_n} \neq \emptyset$, which implies $\beta \cap [B]_{\mathcal{M}_n} \neq \emptyset$. Therefore $\mathcal{M}_n, w \Vdash \Diamond B$. Now assume that $\mathcal{M}_n, w \Vdash \Diamond B$. Then for every $\alpha \in \mathcal{N}_\Diamond(w)$, $\alpha \cap [B]_{\mathcal{M}_n} \neq \emptyset$. Thus for every $\alpha \in \mathcal{N}_k(w)$, $\alpha \cap [B]_{\mathcal{M}_n} \neq \emptyset$, and, by i.h., $\alpha \cap [B]_{\mathcal{M}_k} \neq \emptyset$. Therefore $\mathcal{M}_k, w \Vdash_k \Diamond B$.

Theorem 19 *A formula A is valid in Kojima models for $\mathsf{CCDL^p}$ if and only if it is valid in CINMs for $\mathsf{CCDL^p}$.*

Proof If a Kojima model for $\mathsf{CCDL^p}$ falsifies A, then by Lemma 10 there is a CINM for $\mathsf{CCDL^p}$ that falsifies A. *Vice versa*, if a CINM for $\mathsf{CCDL^p}$ falsifies A, then by Theorem 17 there is a finite CINM for $\mathsf{CCDL^p}$ that falsifies A, and consequently by Lemma 11 there is a Kojima model for $\mathsf{CCDL^p}$ that falsifies A.

Given the previous lemmas and Theorems 4.3 and 4.7 in Kojima [24], we can also see how to obtain an equivalent relational model starting from a CINM for $\mathsf{CCDL^p}$, and *vice versa*. As before, we assume the original CINM to be finite.

Lemma 12 *Let $\mathcal{M}_r = \langle \mathcal{W}, \preceq, \mathcal{R}, \mathcal{V} \rangle$ be a relational model for $\mathsf{CCDL^p}$, and let $\mathscr{R}(w) = \{v \mid w\mathcal{R}v\}$. We define the neighbourhood model $\mathcal{M}_n = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ by taking $\mathcal{W}$, $\preceq$, $\mathcal{V}$ as in $\mathcal{M}_r$, and the following neighbourhood functions:*

$\mathcal{N}_\Box(w) = \{\alpha \subseteq \mathcal{W} \mid \text{for all } v \succeq w, \mathscr{R}(v) \subseteq \alpha\}$;
$\mathcal{N}_\Diamond(w) = \{\alpha \subseteq \mathcal{W} \mid \text{there is } v \succeq w \text{ s.t. } \mathscr{R}(v) \subseteq \alpha\}$.

Then $\mathcal{M}_n$ is a CINM for $\mathsf{CCDL^p}$, and it is pointwise equivalent to $\mathcal{M}_r$.

Lemma 13 *Let $\mathcal{M}_n = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ be a finite CINM for* $\mathsf{CCDL^p}$*. The relational model $\mathcal{M}^* = \langle \mathcal{W}^*, \preceq^*, \mathcal{R}^*, \mathcal{V}^* \rangle$ is defined as follows:*

- $\mathcal{W}^* = \{(w, \alpha) \mid w \in \mathcal{W}, \alpha \in \mathcal{N}_\Diamond(w), \text{ and } \alpha \subseteq \bigcap \mathcal{N}_\Box(w)\}$;
- $(w, \alpha) \preceq^* (v, \beta)$ *iff* $w \preceq v$;
- $(w, \alpha)\mathcal{R}^*(v, \beta)$ *iff* $v \in \alpha$;
- $\mathcal{V}^*((w, \alpha)) = \{p \mid p \in \mathcal{V}(w)\}$ *for all* $w \in \mathcal{W}$.

Then $\mathcal{M}^*$ is a relational model for $\mathsf{CCDL^p}$*. Moreover, for all $A \in \mathcal{L}$ and $w \in \mathcal{W}$, the following claims are equivalent:*

1) $\mathcal{M}_n, w \Vdash A$.
2) For every $(w, \alpha) \in \mathcal{W}^*$, $\mathcal{M}^*, (w, \alpha) \Vdash_r A$.
3) There is $(w, \alpha) \in \mathcal{W}^*$ *such that* $\mathcal{M}^*, (w, \alpha) \Vdash_r A$.

Theorem 20 *A formula A is valid in relational models for* $\mathsf{CCDL^p}$ *if and only if it is valid in CINMs for* $\mathsf{CCDL^p}$*.*

Proof By Lemmas 12 and 13 and Theorem 17. A direct proof of the two lemmas is left to the reader.

7.3.2 Semantic equivalence for CK

We now present the relational models for CK by Mendler and de Paiva [33], and prove directly their equivalence with CINMs. Relational models for CK are defined by enriching Wijesekera's models for $\mathsf{CCDL^p}$ with inconsistent (or "fallible") worlds — *i.e.* worlds satisfying $\bot$ — as follows.

Definition 13 (Relational models for CK) Relational models for CK are defined exactly as relational models for $\mathsf{CCDL^p}$ (Definition 11), except that the standard forcing relation for $\bot$ ($w \nVdash_r \bot$) is replaced by the following ones:

If $w \Vdash_r \bot$, then for every v, $w \preceq v$ or $w\mathcal{R}v$ implies $v \Vdash_r \bot$;
If $w \Vdash_r \bot$, then $w \Vdash_r p$ for every propositional variables $p \in \mathcal{L}$.

Observe that fallible worlds are related through $\preceq$ and $\mathcal{R}$ only to other fallible worlds. Moreover, the above definition preserves the validity of $\top$ and $\bot \supset A$, for all A.

Theorem 21 (Mendler and de Paiva [33]) *The logic CK is sound and complete with respect to relational models for CK.*

As for the case of $\mathsf{CCDL^p}$, we can prove that every relational model can be transformed into an equivalent CINM for CK, and that every *finite* CINM for CK can be transformed into an equivalent relational model. The equivalence of the two semantics is then a consequence of the finite model property of CK with respect to its CINMs. The transformations considered here are relatively similar to those in Lemmas 12 and 13. However in this case they are a bit more complicated because of the presence of inconsistent worlds.

Lemma 14 *Let $\mathcal{M}_r = \langle \mathcal{W}, \preceq, \mathcal{R}, \mathcal{V} \rangle$ be a relational model for* CK. *Moreover, for every $w \in \mathcal{W}$, let $\mathscr{R}(w) = \{v \mid w\mathcal{R}v\}$. We denote with $\mathcal{W}^+$ the set $\{w \in \mathcal{W} \mid \mathcal{M}_r, w \not\Vdash_r \bot\}$ (i.e. the set of consistent worlds of $\mathcal{M}_r$), and for all $\alpha \subseteq \mathcal{W}$, we denote with α^+ the set $\alpha \cap \mathcal{W}^+$.*

We define the neighbourhood model $\mathcal{M}_n = \langle \mathcal{W}^+, \preceq^+, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V}^+ \rangle$, where $\preceq^+$ and $\mathcal{V}^+$ are the restrictions to $\mathcal{W}^+$ of $\preceq$ and $\mathcal{V}$, and $\mathcal{N}_\Box$, $\mathcal{N}_\Diamond$ are the following neighbourhood functions:

$$\mathcal{N}_\Box(w) = \{\alpha^+ \subseteq \mathcal{W} \mid \text{for all } v \succeq w, \mathscr{R}(v) \subseteq \alpha\};$$
$$\mathcal{N}_\Diamond(w) = \{\alpha^+ \subseteq \mathcal{W} \mid \text{there is } v \succeq w \text{ s.t. } \mathscr{R}(v) \subseteq \alpha^+\}.$$

Then $\mathcal{M}_n$ is a CINM for CK. *Moreover, for all $A \in \mathcal{L}$ and $w \in \mathcal{W}^+$,*

$$\mathcal{M}_n, w \Vdash A \quad \textit{iff} \quad \mathcal{M}_r, w \Vdash_r A.$$

Proof It is easy to verify that $\mathcal{M}_n$ is a CINM for CK. In particular, for $WInt$, assume that $\alpha^+ \in \mathcal{N}_\Box(w)$ and $\beta^+ \in \mathcal{N}_\Diamond(w)$. Then there is $v \succeq w$ s.t. $\mathscr{R}(v) \subseteq \beta^+$; thus $\mathscr{R}(v) \subseteq \alpha$. Then $\mathscr{R}(v) \subseteq \alpha \cap \beta^+ = (\alpha \cap \beta)^+$. Therefore $(\alpha \cap \beta)^+ = \alpha^+ \cap \beta^+ \in \mathcal{N}_\Diamond(w)$.

We now prove that for every $w \in \mathcal{W}^+$, $\mathcal{M}_n, w \Vdash A$ if and only if $\mathcal{M}_r, w \Vdash_r A$. This is equivalent to stating that $[A]_{\mathcal{M}_n} = [A]^+_{\mathcal{M}_r}$. As usual, we only consider the modal cases.

$A \equiv \Box B$. Let $w \in \mathcal{W}^+$. $\mathcal{M}_n, w \Vdash \Box B$ iff $[B]_{\mathcal{M}_n} \in \mathcal{N}_\Box(w)$ iff (i.h.) $[B]^+_{\mathcal{M}_r} \in \mathcal{N}_\Box(w)$ iff for all $v \succeq w$, $\mathscr{R}(v) \subseteq [B]_{\mathcal{M}_r}$ iff $\mathcal{M}_r, w \Vdash_r \Box B$.

$A \equiv \Diamond B$. Assume that $\mathcal{M}_r, w \Vdash_r \Diamond B$ and $w \in \mathcal{W}^+$. Then for every $v \succeq w$, there is $u \in \mathcal{W}$ s.t. $v\mathcal{R}u$ and $\mathcal{M}_r, u \Vdash_r B$. Thus for every $v \succeq w$, $\mathcal{R}(v) \not\subseteq \mathcal{W} \setminus [B]_{\mathcal{M}_r}$, which in particular implies that $\mathcal{R}(v) \not\subseteq (\mathcal{W} \setminus [B]_{\mathcal{M}_r})^+$. Moreover, $(\mathcal{W} \setminus [B]_{\mathcal{M}_r})^+ = \mathcal{W}^+ \setminus [B]^+_{\mathcal{M}_r} =$ (i.h.) $\mathcal{W}^+ \setminus [B]_{\mathcal{M}_n}$. Then $\mathcal{W}^+ \setminus [B]_{\mathcal{M}_n} \notin \mathcal{N}_\Diamond(w)$, therefore $\mathcal{M}_n, w \Vdash \Diamond B$. Now assume that $\mathcal{M}_n, w \Vdash \Diamond B$. Then $\mathcal{W}^+ \setminus [B]_{\mathcal{M}_n} \notin \mathcal{N}_\Diamond(w)$. This implies that for every $v \succeq w$, $\mathcal{R}(v) \not\subseteq \mathcal{W}^+ \setminus [B]_{\mathcal{M}_n}$; that is, there is $u \in \mathcal{W}$ s.t. $v\mathcal{R}u$ and $u \notin \mathcal{W}^+ \setminus [B]_{\mathcal{M}_n}$. Thus $u \notin \mathcal{W}^+$ or $u \in [B]_{\mathcal{M}_n}$. If $u \notin \mathcal{W}^+$, then $\mathcal{M}_r, u \Vdash_r \bot$, hence $\mathcal{M}_r, u \Vdash_r B$. If $u \in [B]_{\mathcal{M}_n}$, by inductive hypothesis $u \in [B]^+_{\mathcal{M}_r}$, thus $\mathcal{M}_r, u \Vdash_r B$. Therefore $\mathcal{M}_r, w \Vdash_r \Diamond B$.

Lemma 15 *Let $\mathcal{M}_n = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ be a finite CINM for* CK, *and take $\mathbf{f} \notin \mathcal{W}$. The relational model $\mathcal{M}^* = \langle \mathcal{W}^*, \preceq^*, \mathcal{R}^*, \mathcal{V}^* \rangle$ is defined as follows:*

- $\mathcal{W}^* = \{(w, \alpha) \mid w \in \mathcal{W},\ \mathcal{N}_\Diamond(w) \neq \emptyset,\ \alpha \in \mathcal{N}_\Diamond(w), \text{ and } \alpha \subseteq \bigcap \mathcal{N}_\Box(w)\}$
  $\cup\ \{(v, \bigcap \mathcal{N}_\Box(v) \cup \{\mathbf{f}\}) \mid v \in \mathcal{W} \text{ and } \mathcal{N}_\Diamond(v) = \emptyset\}$
  $\cup\ \{(\mathbf{f}, \{\mathbf{f}\})\}$;
- $(w, \alpha) \preceq^* (v, \beta)$ *iff* $w \preceq v$ *or* $w, v = \mathbf{f}$;
- $(w, \alpha)\mathcal{R}^*(v, \beta)$ *iff* $v \in \alpha$;
- $\mathcal{V}^*((w, \alpha)) = \{p \mid p \in \mathcal{V}(w)\}$ *for all* $w \in \mathcal{W}$; *and* $\mathcal{V}^*((\mathbf{f}, \{\mathbf{f}\})) = Atm$;
- $\mathcal{M}^*, (\mathbf{f}, \{\mathbf{f}\}) \Vdash_r \bot$.

Then $\mathcal{M}^*$ is a relational model for CK. *Moreover, for every $A \in \mathcal{L}$ and $w \in \mathcal{W}$, the following claims are equivalent:*

1) $\mathcal{M}_n, w \Vdash A$.
2) For every $(w,\alpha) \in \mathcal{W}^*$, $\mathcal{M}^*, (w,\alpha) \Vdash_r A$.
3) There is $(w,\alpha) \in \mathcal{W}^*$ *such that* $\mathcal{M}^*, (w,\alpha) \Vdash_r A$.

Proof It is easy to check that $\mathcal{M}^*$ is a relational model for CK, in particular that the conditions on inconsistent worlds are satisfied. We prove by induction on A that 1), 2) and 3) are equivalent. As usual we only consider the inductive cases $A \equiv \Box B$, $\Diamond B$.

- $A \equiv \Box B$.

– 1) implies 2). Assume $\mathcal{M}_n, w \Vdash \Box B$. Then $[B]_{\mathcal{M}_n} \in \mathcal{N}_\Box(w)$, that implies $\bigcap \mathcal{N}_\Box(w) \subseteq [B]_{\mathcal{M}_n}$. Let $(w,\alpha) \in \mathcal{W}^*$, and $(w,\alpha) \preceq^* (v,\beta)$. Then $w \preceq v$, so $\bigcap \mathcal{N}_\Box(v) \subseteq \bigcap \mathcal{N}_\Box(w)$. We distinguish two cases:
  (a) $\mathbf{f} \in \beta$. Then $(v,\beta)\mathcal{R}^*(u,\gamma)$ implies $u \in \bigcap \mathcal{N}_\Box(v)$ or $u = \mathbf{f}$.
  If $u = \mathbf{f}$, then $(u,\gamma) = (\mathbf{f}, \{\mathbf{f}\})$, so $\mathcal{M}^*, (u,\gamma) \Vdash_r B$.
  If $u \in \bigcap \mathcal{N}_\Box(v)$, then $u \in [B]_{\mathcal{M}_n}$. By inductive hypothesis we have $\mathcal{M}^*, (u,\gamma) \Vdash_r B$ for all γ s.t. $(u,\gamma) \in \mathcal{W}^*$.
  (b) $\mathbf{f} \notin \beta$. Then $\beta \subseteq \bigcap \mathcal{N}_\Box(v)$, thus $\beta \subseteq [B]_{\mathcal{M}_n}$. Let $(v,\beta)\mathcal{R}^*(u,\gamma)$. Then $u \in \beta$, so $\mathcal{M}_n, u \Vdash B$. By inductive hypothesis we have $\mathcal{M}^*, (u,\gamma) \Vdash_r B$.
  By (a) and (b) we have that for all $(v,\beta) \succeq^* (w,\alpha)$ and all (u,γ) s.t. $(v,\beta)\mathcal{R}^*(u,\gamma)$, $\mathcal{M}^*, (u,\gamma) \Vdash_r B$. Therefore for all α s.t. $(w,\alpha) \in \mathcal{W}^*$, $\mathcal{M}^*, (w,\alpha) \Vdash_r \Box B$.

– 2) implies 3). Immediate because for every $w \in \mathcal{W}$ there is α such that $(w,\alpha) \in \mathcal{W}^*$.

– 3) implies 1). Assume $\mathcal{M}^*, (w,\alpha) \Vdash_r \Box B$ for an α s.t. $(w,\alpha) \in \mathcal{W}^*$. Then for every $(v,\beta) \succeq^* (w,\alpha)$ and everys (u,γ) s.t. $(v,\beta)\mathcal{R}^*(u,\gamma)$, $\mathcal{M}^*, (u,\gamma) \Vdash_r B$. Thus, in particular, for every δ s.t. $(w,\delta) \in \mathcal{W}^*$, for every (u,γ) s.t. $(w,\delta)\mathcal{R}^*(u,\gamma)$, $\mathcal{M}^*, (u,\gamma) \Vdash_r B$. Take any world $z \in \bigcap \mathcal{N}_\Box(w)$. There exists γ s.t. $(z,\gamma) \in \mathcal{W}^*$. Then $(w, \bigcap \mathcal{N}_\Box(w))\mathcal{R}^*(z,\gamma)$ or $(w, \bigcap \mathcal{N}_\Box(w) \cup \{f\})\mathcal{R}^*(z,\gamma)$ (depending on whether $\mathcal{N}_\Diamond(w) \neq \emptyset$ or $\mathcal{N}_\Diamond(w) = \emptyset$; in the first case $\bigcap \mathcal{N}_\Box(w) \in \mathcal{N}_\Diamond(w)$). Thus $\mathcal{M}^*, (z,\gamma) \Vdash_r B$; and by inductive hypothesis, $\mathcal{M}_n, z \Vdash B$. So $\bigcap \mathcal{N}_\Box(w) \subseteq [B]_{\mathcal{M}_n}$, which implies $[B]_{\mathcal{M}_n} \in \mathcal{N}_\Box(w)$. Therefore $\mathcal{M}_n, w \Vdash \Box B$.

- $A \equiv \Diamond B$.

– 1) implies 2). Assume $\mathcal{M}_n, w \Vdash \Diamond B$, and let $(w,\alpha) \in \mathcal{W}^*$ and $(w,\alpha) \preceq^* (v,\beta)$. We distinguish two cases:
  (a) $\mathbf{f} \in \beta$. Then $(y,\beta)\mathcal{R}^*(\mathbf{f}, \{\mathbf{f}\})$, and $\mathcal{M}^*, (\mathbf{f}, \{\mathbf{f}\}) \Vdash_r B$.
  (b) $\mathbf{f} \notin \beta$. Then $\beta \in \mathcal{N}_\Diamond(y)$, so $\beta \in \mathcal{N}_\Diamond(y)$. By $\mathcal{M}_n, w \Vdash \Diamond B$, we have that for every $\gamma \in \mathcal{N}_\Diamond(w)$, $\gamma \cap [B]_{\mathcal{M}_n} \neq \emptyset$; thus $\beta \cap [B]_{\mathcal{M}_n} \neq \emptyset$. Then there is $u \in \beta$ s.t. $\mathcal{M}_n, u \Vdash B$. By inductive hypothesis, for every δ s.t. $(u,\delta) \in \mathcal{W}^*$, $\mathcal{M}^*, (u,\delta) \Vdash_r B$. Moreover, there is ϵ s.t. $(u,\epsilon) \in \mathcal{W}^*$. Thus $(v,\beta)\mathcal{R}^*(u,\epsilon)$ and $\mathcal{M}^*, (u,\epsilon) \Vdash_r B$.
  By (a) and (b) we have that for every $(v,\beta) \succeq^* (w,\alpha)$, there is (u,γ) s.t. $(v,\beta)\mathcal{R}^*(u,\gamma)$ and $\mathcal{M}^*, (u,\gamma) \Vdash_r B$. Therefore, for every α s.t. $(w,\alpha) \in \mathcal{W}^*$, $\mathcal{M}^*, (w,\alpha) \Vdash_r \Diamond B$.

– 2) implies 3). Immediate because for every $w \in \mathcal{W}$ there is α such that $(w, \alpha) \in \mathcal{W}^*$.
– 3) implies 1). Assume $\mathcal{M}^*, (w, \alpha) \Vdash_r \Diamond B$ for a α s.t. $(w, \alpha) \in \mathcal{W}^*$. Then for every $(v, \beta) \succeq^* (w, \alpha)$, there is (u, γ) s.t. $(v, \beta)\mathcal{R}^*(u, \gamma)$ and $\mathcal{M}^*, (u, \gamma) \Vdash_r B$. Thus in particular, for every δ s.t. $(w, \delta) \in \mathcal{W}^*$, there is (u, γ) s.t. $(w, \delta)\mathcal{R}^*(u, \gamma)$ and $\mathcal{M}^*, (u, \gamma) \Vdash_r B$. We distinguish two cases:
  (*a*) $\mathbf{f} \in \delta$ for a $(w, \delta) \in \mathcal{W}^*$. Then $\mathcal{N}_\Diamond(w) = \emptyset$, so $\mathcal{M}_n, w \Vdash \Diamond B$.
  (*b*) $\mathbf{f} \notin \delta$ for every $(w, \delta) \in \mathcal{W}^*$. Then by inductive hypothesis we have that for every $(w, \delta) \in \mathcal{W}^*$, there is (u, γ) s.t. $(w, \delta)\mathcal{R}^*(u, \gamma)$ and $\mathcal{M}_n, u \Vdash B$. So $u \in \delta$. This means that for every $\delta \in \mathcal{N}_\Diamond(w)$ s.t. $\delta \subseteq \bigcap \mathcal{N}_\Box(w)$, $\delta \cap [B]_{\mathcal{M}_n} \neq \emptyset$. Then by $WInt'$, we have that for every $\epsilon \in \mathcal{N}_\Diamond(w)$, $\epsilon \cap [B]_{\mathcal{M}_n} \neq \emptyset$. Therefore $\mathcal{M}_n, w \Vdash \Diamond B$.

Theorem 22 *A formula A is valid in relational models for* CK *if and only if it is valid in CINMs for* CK*.*

Proof Assume that A is not valid in relational models for CK. Then there are a relational model $\mathcal{M}_r$ and a world w such that $\mathcal{M}_r, w \not\Vdash_r A$. The world w is consistent (*i.e.* $\mathcal{M}_r, w \not\Vdash_r \bot$) as inconsistent worlds satisfy all formulas. Then by Lemma 14, there is a CINM $\mathcal{M}_n$ for CK such that $\mathcal{M}_n, w \not\Vdash A$.

Now, assume that A is not valid in CINMs for CK. Then by Theorem 17, there are a finite model $\mathcal{M}_n$ and a world w such that $\mathcal{M}_n, w \not\Vdash A$. Therefore by Lemma 15, there are a relational model $\mathcal{M}^*$ and a world (w, α) such that $\mathcal{M}^*, (w, \alpha) \not\Vdash_r A$.

8 Further work

The results presented in this article can be extended in several ways, here we highlight some possible directions.

8.1 Non-monotonic systems with $\mathsf{C}_\Box$ and negative interactions a and b

We have shown in Section 4.2 that the combination of rule $\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$ with $\mathsf{neg_a C^{seq}}$ – *i.e.* the generalisation to n principal formulas of rule $\mathsf{neg_a^{seq}}$ – provides a cut-free calculus, and that the admissibility of cut is preserved by the addition of rules $\mathsf{N}_\Diamond^{\mathsf{seq}}$ and $\mathsf{N}_\Box^{\mathsf{seq}}$. In addition, for the corresponding logics we have also provided a Hilbert axiomatisation and a semantic characterisation in terms of CINMs. As we remarked, the addition of a proper generalisation of rule $\mathsf{neg_b^{seq}}$ is by contrast problematic. The rule would be the following

$$\mathsf{neg_b C^{seq}}\ \frac{A_1, ..., A_n, B \Rightarrow \qquad \neg(A_1 \wedge ... \wedge A_n) \Rightarrow B}{\Gamma, \Box A_1, ..., \Box A_n, \Diamond B \Rightarrow C}\ ;$$

but at present it is an open problem whether this rule would give a cut-free calculus.

Alternatively, one could consider the rule

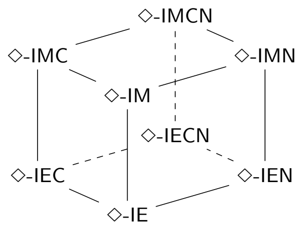

Fig. 10: Extended lattice of $\Diamond$-logics.

$$\mathsf{neg_b C'^{seq}} \ \frac{A_1, ..., A_n, B \Rightarrow \qquad \neg A_1 \Rightarrow B \quad ... \quad \neg A_n \Rightarrow B}{\Gamma, \Box A_1, ..., \Box A_n, \Diamond B \Rightarrow C} \ .$$

It can be shown that this rule gives a cut-free calculus. However, since $\neg A_1 \lor ... \lor \neg A_n$ is not intuitionistically equivalent to $\neg(A_1 \land ... \land A_n)$, it is not obvious how to extend the Hilbert axiomatisation and the semantic characterisation to the resulting logic. The addition of this rule would be natural in a non-normal modal extension of a suitable intermediate logic. The whole issue will be object of future work.

8.2 Systems containing $\mathsf{C}_\Diamond$

In this paper we have restricted the analysis to systems not containing axiom $\mathsf{C}_\Diamond$. This axiom is of particular significance in the intuitionistic context since it can be seen as a cut-off point between the constructive and the intuitionistic tradition. In future work we aim to extend our framework to cover also such systems, here we limit ourselves to some preliminary remarks.

If we restrict our analysis to systems without interaction between the modalities, a semantic characterisation of axiom $\mathsf{C}_\Diamond$ can be given by requiring that $\mathcal{N}_\Diamond$ is closed under intersection. To make this precise, let us extend the $\Diamond$-family to the systems containing $\mathsf{C}_\Diamond$: $\Diamond$-logics are now defined by adding to IPL the congruence rule $\mathsf{E}_\Diamond$ and any combination of axioms $\mathsf{M}_\Diamond$, $\mathsf{N}_\Diamond$, and $\mathsf{C}_\Diamond$. We obtain the picture in Figure 10, which contains 8 non equivalent systems.

In order to prove completeness we have to modify Definition 6 and Definition 7 of canonical models as follows.

Definition 14 (Canonical models for non-monotonic $\Diamond$-systems) Let L be any $\Diamond$-system not containing axiom $\mathsf{M}_\Diamond$. The *canonical model* $\mathcal{M}^c = \langle \mathcal{W}^c, \preceq^c, \mathcal{N}_\Diamond^c, \mathcal{V}^c \rangle$ for L is defined as in Definition 6, except that it does not contain $\mathcal{N}_\Box^c$, and $\mathcal{N}_\Diamond^c$ is now defined as

$$\mathcal{N}_\Diamond^c(X) = \{\mathcal{W}^c \setminus \uparrow_{pr} A \mid \Diamond A \notin X\}.$$

Definition 15 (Canonical models for monotonic $\Diamond$-systems) Let L be any $\Diamond$-system containing axiom $\mathsf{M}_\Diamond$. The *canonical model* $\mathcal{M}^c = \langle \mathcal{W}^c, \preceq^c, \mathcal{N}_\Diamond^+, \mathcal{V}^c \rangle$ for L is defined as in Definition 15, except for $\mathcal{N}_\Diamond^+$, which is

$$\mathcal{N}_{\Diamond}^{+}(X) = \{\alpha \subseteq \mathcal{W}^c \mid \text{there is } A \in \mathcal{L} \text{ s.t. } \Diamond A \notin X \text{ and } \alpha \subseteq \mathcal{W}^c \setminus \uparrow_{pr} A\}.$$

Theorem 23 *Every $\Diamond$-logic is sound and complete with respect to the corresponding CINMs.*

Proof As usual, the claim follows from the truth lemma: $X \Vdash A$ if and only if $A \in X$, and the fact that if L contains $\mathsf{M}_{\Diamond}$, $\mathsf{N}_{\Diamond}$, or $\mathsf{C}_{\Diamond}$, then the canonical model for L satisfies the corresponding semantic conditions. We only show a sketch of the proof considering canonical models for non-monotonic systems.

For the truth lemma we only address the case $A = \Diamond B$. Assume $\Diamond B \notin X$. Then $\mathcal{W}^c \setminus \uparrow_{pr} B \in \mathcal{N}_{\Diamond}^{c}(X)$, and by inductive hypothesis $\mathcal{W}^c \setminus [B] \in \mathcal{N}_{\Diamond}^{c}(X)$, thus $X \not\Vdash \Diamond B$. Now assume $X \not\Vdash \Diamond B$. Then $\mathcal{W}^c \setminus [B] \in \mathcal{N}_{\Diamond}^{c}(X)$, and by inductive hypothesis $\mathcal{W}^c \setminus \uparrow_{pr} B \in \mathcal{N}_{\Diamond}^{c}(X)$. By definition, there is $C \in \mathcal{L}$ such that $\Diamond C \notin X$ and $\uparrow_{pr} C = \uparrow_{pr} B$. Thus $\vdash C \supset\subset B$, and by $\mathsf{E}_{\Diamond}$, $\vdash \Diamond C \supset\subset \Diamond B$. It follows from the properties of prime sets that $\Diamond B \notin X$.

Now we show that $\mathcal{N}_{\Diamond}^{c}$ is closed under intersection if L contains $\mathsf{C}_{\Diamond}$. Assume $\alpha, \beta \in \mathcal{N}_{\Diamond}^{c}(X)$. Then there are $A, B \in \mathcal{L}$ such that $\alpha = \mathcal{W}^c \setminus \uparrow_{pr} A$, $\beta = \mathcal{W}^c \setminus \uparrow_{pr} B$, and $\Diamond A, \Diamond B \notin X$. Since X contains $\mathsf{C}_{\Diamond}$ and is closed under derivation, $\Diamond(A \vee B) \notin X$. Thus $\mathcal{W}^c \setminus \uparrow_{pr}(A \vee B) \notin \mathcal{N}_{\Diamond}^{c}(X)$, where $\mathcal{W}^c \setminus \uparrow_{pr}(A \vee B) = \mathcal{W}^c \setminus (\uparrow_{pr} A \cup \uparrow_{pr} B) = (\mathcal{W}^c \setminus \uparrow_{pr} A) \cap (\mathcal{W}^c \setminus \uparrow_{pr} B) = \alpha \cap \beta$.

This result can be extended to logics with both $\Box$ and $\Diamond$ but without interactions between the modalities. On the contrary, as a consequence of the modification of the definition of canonical models, for logics with interactions between $\Box$ and $\Diamond$ the completeness proofs presented in Section 6 do not work anymore. Further investigation is required to establish whether in presence of $\mathsf{C}_{\Diamond}$ we can preserve the semantic conditions connecting $\mathcal{N}_{\Box}$ and $\mathcal{N}_{\Diamond}$ that we considered in this work, or whether we need to consider different connections instead.

From the point of view of sequent calculi, $\Diamond$-logics containing $\mathsf{C}_{\Diamond}$ could be covered by modifying rules $\mathsf{E}_{\Diamond}^{\mathsf{seq}}$, $\mathsf{M}_{\Diamond}^{\mathsf{seq}}$, and $\mathsf{N}_{\Diamond}^{\mathsf{seq}}$ in the following way, where Δ is a multiset of formulas of $\mathcal{L}$.

$$\mathsf{E}_{\Diamond}\mathsf{C}^{\mathsf{seq}} \ \frac{A \Rightarrow B_1, ..., B_n \qquad B_1 \Rightarrow A \quad ... \quad B_1 \Rightarrow A}{\Gamma, \Diamond A \Rightarrow \Diamond B_1, ..., \Diamond B_n, \Delta}$$

$$\mathsf{M}_{\Diamond}\mathsf{C}^{\mathsf{seq}} \ \frac{A \Rightarrow B_1, ..., B_n}{\Gamma, \Diamond A \Rightarrow \Diamond B_1, ..., \Diamond B_n, \Delta} \qquad \mathsf{N}_{\Diamond}\mathsf{C}^{\mathsf{seq}} \ \frac{A \Rightarrow}{\Gamma, \Diamond A \Rightarrow \Delta}$$

If compared with the other rules considered in this work, rules for $\mathsf{C}_{\Diamond}$ have the crucial difference of containing multiple formulas on the right-hand side of sequents. In order to admit these rules we have to take as base calculus instead of G3ip a multisuccedent calculus for intuitionistic logic, as for instance the propositional fragment of m-G3i in Troelstra and Schwichtenberg [41] (let us call it m-G3ip). Then sequent calculi for $\Diamond$-systems containing $\mathsf{C}_{\Box}$ are defined by extending m-G3ip as follows:

G.$\Diamond$-IEC	:=	$\mathsf{E}_{\Diamond}\mathsf{C}^{\mathsf{seq}}$
G.$\Diamond$-IMC	:=	$\mathsf{M}_{\Diamond}\mathsf{C}^{\mathsf{seq}}$
G.$\Diamond$-IECN	:=	$\mathsf{E}_{\Diamond}\mathsf{C}^{\mathsf{seq}} + \mathsf{N}_{\Diamond}^{\mathsf{seq}}$
G.$\Diamond$-IMCN	:=	$\mathsf{M}_{\Diamond}\mathsf{C}^{\mathsf{seq}} + \mathsf{N}_{\Diamond}^{\mathsf{seq}}$

We can prove the following theorem.

Theorem 24 *The cut rule*

$$\mathsf{cut}\ \dfrac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

is admissible in G.$\Diamond$-IEC*,* G.$\Diamond$-IMC*,* G.$\Diamond$-IECN*, and* G.$\Diamond$-IMCN*.*

As before, the proof of cut admissibility goes through the admissibility of contraction and consists in showing how to remove any application of cut in a derivation. Here we only show two significant cases in which the cut formula is principal in the last rule applied in the derivation of both premisses of cut.

- ($\mathsf{E}_\Diamond\mathsf{C}^{\mathsf{seq}}$; $\mathsf{E}_\Diamond\mathsf{C}^{\mathsf{seq}}$). Let $\Gamma_1 = B_1, ..., B_n$ and $\Gamma_2 = D_1, ..., D_m$.

$$\dfrac{\mathsf{E}_\Diamond\mathsf{C}^{\mathsf{seq}}\ \dfrac{A \Rightarrow \Gamma_1, C \quad B_1 \Rightarrow A \ \dots \ B_n \Rightarrow A \quad C \Rightarrow A}{\Gamma, \Diamond A \Rightarrow \Diamond\Gamma_1, \Diamond C, \Delta} \qquad \dfrac{C \Rightarrow \Gamma_2 \quad D_1 \Rightarrow C \ \dots \ D_m \Rightarrow C}{\Gamma', \Diamond C \Rightarrow \Diamond\Gamma_2, \Delta'}\ \mathsf{E}_\Diamond\mathsf{C}^{\mathsf{seq}}}{\Gamma, \Gamma', \Diamond A \Rightarrow \Diamond\Gamma_1, \Diamond\Gamma_2, \Delta, \Delta'}\ \mathsf{cut}$$

$$\rightsquigarrow$$

$$\mathsf{E}_\Diamond\mathsf{C}^{\mathsf{seq}}\ \dfrac{\mathsf{cut}\ \dfrac{A \Rightarrow \Gamma_1, C \qquad C \Rightarrow \Gamma_2}{A \Rightarrow \Gamma_1, \Gamma_2} \qquad B_1 \Rightarrow A \ \dots \ B_n \Rightarrow A \qquad \left(\mathsf{cut}\ \dfrac{D_i \Rightarrow C \qquad C \Rightarrow A}{D_i \Rightarrow A}\right)_{i=1}^{m}}{\Gamma, \Gamma', \Diamond A \Rightarrow \Diamond\Gamma_1, \Diamond\Gamma_2, \Delta, \Delta'}$$

- ($\mathsf{E}_\Diamond\mathsf{C}^{\mathsf{seq}}$; $\mathsf{N}_\Diamond\mathsf{C}^{\mathsf{seq}}$). Let $\Gamma_1 = B_1, ..., B_n$.

$$\dfrac{\mathsf{E}_\Diamond\mathsf{C}^{\mathsf{seq}}\ \dfrac{A \Rightarrow \Gamma_1, C \quad B_1 \Rightarrow A \ \dots \ B_n \Rightarrow A \quad C \Rightarrow A}{\Gamma, \Diamond A \Rightarrow \Diamond\Gamma_1, \Diamond C, \Delta} \qquad \dfrac{C \Rightarrow}{\Gamma', \Diamond C \Rightarrow \Delta'}\ \mathsf{N}_\Diamond\mathsf{C}^{\mathsf{seq}}}{\Gamma, \Gamma', \Diamond A \Rightarrow \Diamond\Gamma_1, \Diamond\Gamma_2, \Delta, \Delta'}\ \mathsf{cut}$$

$$\rightsquigarrow$$

$$\mathsf{E}_\Diamond\mathsf{C}^{\mathsf{seq}}\ \dfrac{\mathsf{cut}\ \dfrac{A \Rightarrow \Gamma_1, C \qquad C \Rightarrow}{A \Rightarrow \Gamma_1} \qquad B_1 \Rightarrow A \ \dots \ B_n \Rightarrow A}{\Gamma, \Gamma', \Diamond A \Rightarrow \Diamond\Gamma_1, \Diamond\Gamma_2, \Delta, \Delta'}$$

Again, this result can be extended to logics with $\Box$ and $\Diamond$ but without interactions (after rewriting the rules for $\Box$ in their multi-succedent versions). We leave to future work the investigation of interactions between the modalities which in presence of $\mathsf{C}_\Diamond$ give both cut-free calculi and a satisfactory characterisation in terms of CINMs.

8.3 Combinations of monotonic and non-monotonic modalities

In all considered systems $\Box$ and $\Diamond$ are either both monotonic or both non-monotonic. However from a combinatorial perspective, and possibly under certain interpretation of the modalities, it makes sense to consider the cases in which one modality is monotonic and the other one is non-monotonic.

Let us consider first the cases in which one modality is characterised only by the congruence rule and the other modality is characterised only by the monotonicity rule (*i.e.* there are no axioms for N and C). It can be shown that the only rule for interaction which gives a cut-free calculus is

$$\mathsf{str}^{\mathsf{seq}} \; \frac{A, B \Rightarrow}{\Gamma, \Box A, \Diamond B \Rightarrow C} \; .$$

In calculi defined by adding the other two interactions ($\mathsf{weak}_{\mathsf{a}}^{\mathsf{seq}}$ + $\mathsf{weak}_{\mathsf{b}}^{\mathsf{seq}}$ and $\mathsf{neg}_{\mathsf{b}}^{\mathsf{seq}}$ + $\mathsf{neg}_{\mathsf{a}}^{\mathsf{seq}}$) the cut rule is not admissible and it is possible to find counterexamples to cut elimination. For instance, the counterexample presented in Example 3 still holds when $\Diamond$ is non-monotonic. Concerning extensions with rules for N and C, we remark that $\mathsf{weak}_{\mathsf{a}}^{\mathsf{seq}}$ is derivable from $\mathsf{N}_{\Diamond}^{\mathsf{seq}}$, whence in principle there might be more combinations of rules enjoying cut admissibility. We leave to future investigation the study of these combinations and the semantic properties of the resulting systems.

8.4 Further topics

Our results can be extended in other directions. First of all, we can study further extensions of the lattice of intuitionistic non-normal modal logics by axioms analogous to the standard modal ones such as T, D, $\mathsf{4}$, $\mathsf{5}$, *etc.* In the literature there are several proposals of proof systems for extensions of classical non-normal modal logics on the one hand, and of constructive modal logics on the other by the axioms T, D, $\mathsf{4}$, $\mathsf{5}$, *etc*: sequent calculi for the classical cube extended with these axioms have been studied in Indrzejczak [22,23], Lellmann and Pimentel [27], and Orlandelli [29], whereas nested sequent calculi for analogous extensions of CK have been proposed in Arisaka *et al.* [2]. On the semantical side, neighbourhood semantics for some intuitionistic monomodal logics containing T have been recently considered by Witczak [45].

Furthermore, we can study computational and proof-theoretical properties such as complexity bounds and interpolation. Concerning the latter, in Orlandelli [29] a constructive proof of Craig interpolation for the cube of classical non-normal modal logics is given basing on suitable sequent calculi; moreover Iemhoff [21] proposes a set of general conditions on the form of sequent rules ensuring uniform interpolation in modal logics. Furthermore, the simple format of Gentzen calculi presented in this work is not the most adequate for proof search and countermodel extraction, in this respect we would like to develop sequent calculi with invertible rules and that allow for direct countermodel extraction. Such calculi will likely have a more complex structure, like labelled, nested or hypersequent calculi (invertible labelled calculi for classical non-normal modal logics allowing for direct countermodel extraction are presented in Negri [28] and in Dalmonte *et al.* [8]). Finally, it would be interesting to see whether these logics, similarly to CK (Bellin *et al.* [4]), can be given a type-theoretical interpretation by a suitable extension of the typed lambda-calculus. All of this will be part of our future research.

9 Conclusion

This article represents the initial step towards a general investigation of non-normal modalities with an intuitionistic base. We have defined a new family of intuitionistic non-normal modal logics that can be seen as intuitionistic counterparts of classical non-normal modal logics. In particular, we have defined 12 monomodal logics – 8 logics with $\Box$ modality and 4 logics with $\Diamond$ modality – and 24 bimodal logics. For each of them we have provided both a Hilbert axiomatisation and a cut-free sequent calculus. All logics are decidable and contain some of the modal axioms characterising the classical cube. In addition, bimodal logics contain interactions between the modalities that can be seen as "weak duality principles", and express under which conditions two formulas $\Box A$ and $\Diamond B$ are jointly inconsistent. On the basis of the different strength of such interactions, we identify different intuitionistic counterparts of a given classical logic. The picture we get is richer than in the classical case, as logics which collapse in the classical setting are distinct in the intuitionistic one: whereas the classical cube contains 8 logics, the intuitionistic bimodal lattice features 24.

Next, we have given a modular semantic characterisation of the logics by means of Coupled Intuitionistic Neighbourhood Models. The models contain an order relation and two neighbourhood functions handling the modalities separately. For the two functions we consider the standard properties of neighbourhood models, moreover they can be combined in different ways reflecting the possible interactions between $\Box$ and $\Diamond$. Through a filtration argument we have also proved that all logics enjoy the finite model property. Our semantics turned out to be a versatile tool to analyse intuitionistic non-normal modal logics, which is capable of capturing further well-known logics such as Constructive K and the propositional fragment of Wijesekera's CCDL.

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