# Hypersequent calculi for non-normal modal and deontic logics: Countermodels and optimal complexity*

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**Abstract**

We present some hypersequent calculi for all systems of the classical cube and their extensions with axioms $T$, $P$, $D$, and, for every $n \geq 1$, rule $RD_n^+$. The calculi are internal as they only employ the language of the logic, plus additional structural connectives. We show that the calculi are complete with respect to the corresponding axiomatisation by a syntactic proof of cut elimination. Then, we define a terminating proof search strategy in the hypersequent calculi and show that it is optimal for coNP-complete logics. Moreover, we show that from every failed proof of a formula or hypersequent it is possible to directly extract a countermodel of it in the bi-neighbourhood semantics of polynomial size for coNP logics, and for regular logics also in the relational semantics. We finish the paper by giving a translation between hypersequent *rule applications* and *derivations* in a labelled system for the classical cube.

**Keywords** Non-normal modal logic, deontic logic, hypersequent calculus, neighbourhood semantics, countermodels, optimal complexity.

## 1 Introduction

Non-normal modal logics–NNMLs for short–have a long history, going back to the seminal works by Kripke, Montague, Segeberg, Scott, and Chellas (see [4] for an introduction). They are "non-normal" as they do not contain all axioms

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of minimal normal modal logic **K**. NNMLs find an interest in several areas, from epistemic to deontic reasoning. They also play a role in multi-agent reasoning and strategic reasoning in games. For instance in epistemic reasoning they offer a simple (although partial) solution to the problem of logical omniscience (see [39]); in deontic logic, they allow avoiding well-known paradoxes (such as Ross's Paradox) and to represent conflicting obligations (see [19]); multi-agent logics with non-normal modalities have been proposed to capture agency and ability: $\Box A$ is read as "the agent can bring about $A$" (see [11]); a related interpretation is the game-theoretical interpretation of $\Box A$ as "the agent has a winning strategy to bring about $A$" (indeed, non-normal monotonic logic **M** can be seen as a 2-agent case of coalition logic with determinacy [34]). Finally, NNMLs are needed also when $\Box A$ is interpreted as "$A$ is true in most of the cases" [1].

In this work, we consider the *classical cube* of NMMLs, given by the extensions of minimal modal logic **E**, containing only the *congruence rule RE*, with axioms $C$, $M$ and $N$ (cf. Figure 1). We also consider extensions with axioms/rules $T$, $D$, $P$, and $RD_n^+$, where $T$ is the *reflexivity* axiom in classical normal modal logic, and the others axioms are significant in deontic logic. More precisely, reading $\Box A$ as "it is obligatory that $A$", $D$ is the characteristic axiom of deontic logic $\neg(\Box A \wedge \Box\neg A)$, expressing that something and its negation cannot at the same time be obligatory; and $P$ is the axiom $\neg\Box\bot$, expressing that something impossible cannot be obligatory. Although the axioms $P$ and $D$ are equivalent in normal modal logic, this does not hold in the non-normal setting. The system **MNP** is considered as a meaningful *minimal system of Deontic Logic* [19, 31]. Finally, although the rules $RD_n^+$ have never been considered "officially" in the literature, (but see [19] and [20]), they properly generalise the axiom $D$ for systems without $C$, expressing that there cannot be $n$ incompatible obligations: if $\neg(A_1 \wedge \ldots \wedge A_n)$ then $\neg(\Box A_1 \wedge \ldots \wedge \Box A_n)$.

NNMLs have a well-understood semantics defined in terms of neighbourhood models [4, 32]: in these models each world $w$ has an associated set of neighbourhoods $\mathcal{N}(w)$, each one of them being a set of worlds/states. If we accept the traditional interpretation of a "proposition" as a set of worlds (= its truth set), we can think of each neighbourhood in $\mathcal{N}(w)$ as the proposition: a formula $\Box A$ is true in a world $w$ if "the proposition" $A$, i.e., the truth-set of $A$, belongs to $\mathcal{N}(w)$. The classical cube and all mentioned extensions can be modelled by imposing additional closure properties of the set of neighbourhoods.

In this work we adopt a variant of the neighbourhood semantics defined in terms of bi-neighbourhood models [9]: in these structures each world has associated a set of *pairs* of neighbourhoods. The intuition is that the two components of a pair provide positive and negative support for a modal formula. This variant is significant and more natural for "non-monotonic" logics (i.e., not containing axiom $M$). The reason is that, instead of specifying exactly the truth sets in $\mathcal{N}(w)$, the pairs of neighbourhoods specify *lower* and *upper* bounds of truth sets, so that the same pair may be a "witness" for several propositions. For this reason, the generation of *countermodels*, one of the goals of the present work, is easier in the bi-neighbourhood semantics than in the standard one. Bi-neighbourhood models can be transformed into standard ones and vice-versa.

The proof-theory of NNMLs is not quite as developed as their semantics, apart from early works on regular modal logics like [12]. In particular, it is curious to note that, although some proof-systems for NNMLs have been proposed in the past, countermodel generation has been rarely addressed and computational

properties of proof systems are seldom analysed. Indeed, the works [25, 15, 29, 9] propose countermodel extraction in the neighbourhood semantics, but all of them require either a complicated procedure or an extended language with labels. The recent [26] presents a nested sequent calculus for a logic combining normal and monotone non-normal modal logic that supports countermodel extraction. However the nested sequent structure is not suitable for logics lacking monotonicity. In contrast, cut-free sequent/linear nested calculi for the classical cube and its extensions with standard axioms of normal modal logics (the non-normal counterpart of logics from **K** to **S5**), including deontic axioms $D$ and $P$, are studied in [21, 22, 27, 30]. In particular, [30] focuses on cut-free sequent calculi on calculi for deontic logic, partially covering the family of systems defined in this paper. However, neither semantic completeness, nor countermodel extraction, nor complexity are studied in the mentioned papers.

In this work, we intend to fill this gap by proposing *modular* calculi for the classical cube and the mentioned deontic extensions that provide *direct countermodel extraction* and are of *optimal complexity.* Our calculi are semantically based on bi-neighbourhood models, and have two syntactic features: they manipulate hypersequents and sequents may contain blocks of formulas in the antecedent. A hypersequent [2] is just a multiset of sequents and can be understood as a (meta-logical) disjunction of sequents. Sequents within hypersequents can be read as formulas of the logic and, for this reason, our calculi are "almost" internal. In particular, each block is interpreted as a formula $\Box A$, where $A$ is the conjunction of the formulas occurring in the block. Intuitively, each block represents a neighbourhood satisfying one or more boxed formulas, and this allows for the formulation of modular calculi. It is worth noticing that the calculi have also good proof-theoretical properties, as they support a syntactic proof of cut admissibility.

We make clear that, for the purpose of having sound and complete calculi for NNMLs, neither hypersequents, nor blocks are necessary, as for instance the sequent calculi in [25, 30, 21, 22] show. But as we shall see, the hypersequent framework is very adequate to extract countermodels from a single failed proof, ensuring at the same time good computational and structural properties. As a matter of fact, even in the bi-neighbourhood semantics, non-normal modal logics, in particular without monotonicity, ultimately need to consider truth sets of formulas. Hence, in order to obtain calculi suitable for a reasonably straightforward countermodel construction, we need to be able to represent essentially all worlds of a possible model in the data structure used by the calculus. While this could also be accomplished by other types of calculi, for obtaining *small* countermodels in non-monotonic logics it is crucial that every world (represented by a component of the hypersequent) has access to *all* other worlds which have been constructed so far. This very strongly suggests a *flat* structure, as given by hypersequents, in contrast for instance, with the tree-like structure of nested sequents.

A further advantage of using hypersequents is that all rules become invertible, thus there is no need for backtracking in proof search. For the same reason, the hypersequent calculi provide *direct* countermodel extraction: from *one* failed proof we can *directly* extract a countermodel in the bi-neighbourhood semantics of the sequent/formula at the root of the derivation. A particular case is the one of regular logics, i.e., logics containing both axioms $M$ and $C$ (whence normal modal logic **K** as well). These systems admit a relational semantics. We

show how to extract a relational countermodel from a failed proof-search in the calculi for these logics as well.

We also consider the problem of obtaining optimal decision procedures. The known complexity bounds are not the same for logics without and with axiom $C$. Namely (see [39]), the former are coNP, whereas the latter are PSPACE, a fact that also follows by a general result on non-iterative modal logics [36]. For logics with $C$ (belonging to our cube), a PSPACE decision procedure can be obtained by standard proof-search in sequent calculi, like those ones in [25]. Therefore we concentrate on the more significant case of logics *without* $C$: it turns out that for these logics our calculi provide an *optimal* coNP decision procedure. For logics including $C$, we can still obtain an optimal PSPACE decision procedure by adopting an *un-Kleene'd* version of our calculi, which "sacrifices" the invertibility of rules.

We finish this work by presenting a formal translation between hypersequent calculi, restricted to the classical cube, and the labelled calculi $\mathbf{LSE}^*$, presented in [9]. As mentioned above, our calculi have an internal flavour, since sequents have an interpretation within the logics (although hypersequents do not). Labelled systems, on the other hand, are intrinsically external due to the use of symbols that are not in the base logical language, in the form of labels. Establishing translations between sequent based systems and labelled systems is often a hard task [38, 6, 17]. Indeed, hyper/nested sequents typically carry the semantical information within their structure, while labels explicitly mark semantical behaviours to formulas. The results presented in this work shows that our hypersequent calculi provide a compact encoding of derivations in the labelled framework.

All in all, we believe that the structure of our calculi, namely hypersequents with blocks, is adequate for NNMLs from a semantical, computational and a proof-theoretical point of view since it: (i) has a semantic interpretation; (ii) allows direct countermodel generation; (iii) supports optimal complexity decision procedures; (iv) has good proof-theoretical properties; and (v) has a natural translation to labelled systems present in the literature.

This article is a thoroughly revised and significantly extended version of the conference paper [8]. Some of the most significant extensions with respect to that work are the modular extension of all the results to twice the amount of axioms, the extension to the relational semantics for regular logics, and the investigation of the formal relation with the calculi presented in [9], in the form of mutual simulation.

The plan of the paper is as follows: In Section 2 we introduce the logical systems considered in this work. In Section 3, we present both standard neighbourhood semantics and its bi-neighbourhood variant. In Section 4, we introduce hypersequent calculi and we prove the main proof-theoretical properties, including cut-admissibility, from which also follows their syntactic completeness. In Section 5, we show how the calculi can provide a decision procedure for the respective logics and we analyse their complexity. In Section 6, we show how the calculi can be used to extract directly countermodels from failed proofs, which is one of the main goals of this work; additionally, this directly yields semantic completeness. Finally in Section 7 we explore the relation of hypersequent calculi with previously introduced labelled calculi for the classical cube of NNMLs, while Section 8 contains some final discussion and conclusions.

| | | | | | |
|---|---|---|---|---|---|
| $RE$ | $\dfrac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$ | $RD_n^+$ | $\dfrac{\neg(A_1 \wedge ... \wedge A_n)}{\neg(\Box A_1 \wedge ... \wedge \Box A_n)}$ $(n \geq 1)$ | | |
| $M$ | $\Box(A \wedge B) \to \Box A$ | $C$ | $\Box A \wedge \Box B \to \Box(A \wedge B)$ | $N$ | $\Box\top$ |
| $T$ | $\Box A \to A$ | $D$ | $\neg(\Box A \wedge \Box\neg A)$ | $P$ | $\neg\Box\bot$ |

**Figure 1:** Modal axioms and rules.

## 2 Non-normal modal logics as axiom systems

In this section we introduce, axiomatically, the class of non-normal modal logics we consider in this work. We observe that, throughout the text, "axiom" will be used as short for "axiom scheme".

**Definition 2.1.** *Non-normal modal logics* are defined over a propositional modal language $\mathcal{L}$, based on a set $Atm = \{p_1, p_2, p_3, ...\}$ of countably many propositional variables. The well-formed *formulas* of $\mathcal{L}$ are defined by the following grammar

$$A ::= p \mid \bot \mid \top \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A.$$

In the following, we use $A, B, C, D, E$ and $p, q, r$ as metavariables for, respectively, arbitrary formulas and atoms of $\mathcal{L}$. We consider the standard abbreviations $\neg A := A \to \bot$, $A \leftrightarrow B := (A \to B) \wedge (B \to A)$, and $\Diamond A := \neg\Box\neg A$.

**Definition 2.2.** The family of non-normal modal logics is generated by

1. any axiomatization of classical propositional logic (**CPL**) formulated in the language $\mathcal{L}$, comprising the rule of *modus ponens* ($MP$)

$$MP\ \frac{A \to B \qquad A}{B}$$

2. The rule $RE$ of Figure 1.

3. Any or none of the other axioms or rules of Figure 1.

The minimal non-normal modal logic is **E**, defined by only items 1 and 2 above. We denote non-normal modal logics by **EX**, where **X** stands for the (possibly empty) additional set of axioms and rules from Figure 1. We adopt the convention of replacing E with M for systems containing axiom M, which are consequently denoted by **MX**. We also drop the "R" of rule $RD_n^+$. E.g., we write $\mathbf{MD_3^+}$ for the logic given by extending **E** with the axiom $M$ and rule $RD_3^+$. In addition, given a non-normal modal logic **L**, we will write $\mathbf{L}^*$ to indicate an arbitrary extension of **L** with some other axioms.

As usual, we say that a formula $A$ is *provable* in **L** (denoted by $\vdash_{\mathbf{L}} A$) if it is an instance of an axiom of **L** or it is obtained from previous formulas by applying the rules of **L**.

Logic **E** is the weakest system of the so-called *classical cube* [4, 27], generated by any combination of axioms $M$, $C$, and $N$, as shown in Figure 2.

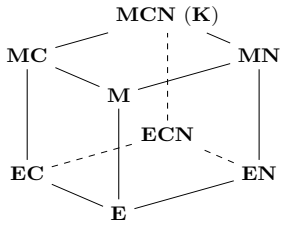


**Figure 2:** The classical cube.

As it is well known, the axioms $M$ and $N$ are respectively equivalent to the rules of *monotonicity* $RM$ and *necessitation* $RN$

$$RM\ \frac{A \to B}{\Box A \to \Box B} \qquad RN\ \frac{A}{\Box A}$$

Moreover, axiom $K$: $\Box(A \to B) \to \Box A \to \Box B$ is derivable in **MC**. As a consequence, the top system **MCN** coincides with the minimal normal modal logic **K**.

In the following, we call *monotonic* any system containing axiom $M$ (and *non-monotonic* otherwise), we call *regular* any system containing both axioms $M$ and $C$, and we call *normal* any system containing $M$, $C$, and $N$.

In the present work, we will in particular consider extensions of the basic classical cube with any combination of the axioms $T$, $D$, $P$, and for every $n \in \mathbb{N}$, $n \geq 1$, the rule $RD_n^+$ (Figure 2).

The logics given by the rules $RD_n^+$ have a peculiar interest in deontic logic since the latter exclude the possiblity of having $n$ obligations that cannot be realised all together. Let us consider the following example, essentially from Hansson [20, p. 41]: (1) I have to keep my mobile switched on (as I'm waiting for an urgent call), (2) I have to attend my child schoolplay, (3) being in the audience of a schoolplay I must keep my mobile switched off. Representing these three claims by *mobile_on*, *schoolplay*, and $\neg(mobile_on \wedge schoolplay)$, by using rule $RD_3^+$, it can be concluded that the three obligations are incompatible:

$$\neg(\Box mobile_on \wedge \Box schoolplay \wedge \Box\neg(mobile_on \wedge schoolplay))$$

This conclusion cannot be obtained in any non-normal modal logic without $RD_3^+$ or $C$, even if it contains both deontic axioms $D$ and $P$.

While rules $RD_n^+$ are entailed by axioms $D$ and $P$ in normal modal logics, this is not the case in non-normal modal logics, therefore they may be considered explicitly. It is also worth noting that the axioms $D$ and $P$ are equivalent in normal modal logics, but not in non-normal ones.

The relations among the systems defined by adding $D$, $P$, and $RD_n^+$ to the systems of the classical cube are displayed in Figure 3. Finally, observe also that all $D$, $P$, and $RD_n^+$ are entailed by axiom $T$.

# 3 Semantics

The standard semantic characterisation of non-normal modal logics is given in terms of *neighbourhood* models [32], also called *minimal* [4], or *Scott-Montague*.

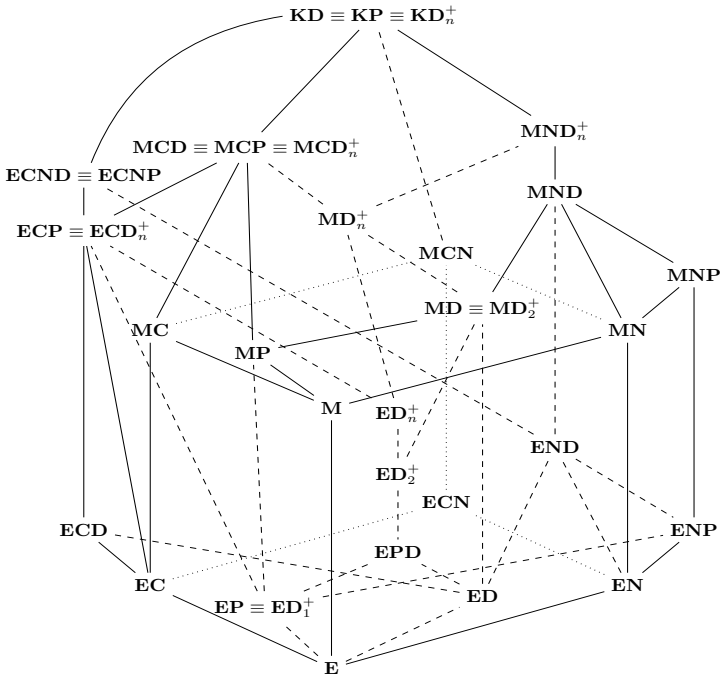


**Figure 3:** Diagram of deontic systems ("Pantheon").

In this work they will be called *standard neighbourhood*, or just *standard* models. Standard neighbourhood models are a generalisation of Kripke models for normal modal logics, where the binary relation is replaced by a so-called neighbourhood function, which assigns to each world a set of sets of worlds. Intuitively, the neighbourhood function assigns to each world the propositions that are necessary/obligatory/etc. in it. Standard neighbourhood models are defined as follows.

**Definition 3.1** (Standard Neighbourhood Model)**.** A *standard neighbourhood model* is a triple $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set, whose elements are called worlds, $\mathcal{N}$ is a function $\mathcal{W} \longrightarrow \mathcal{P}\mathcal{P}(\mathcal{W})$, called neighbourhood function, and $\mathcal{V} : Atm \longrightarrow \mathcal{P}(\mathcal{W})$ is a valuation function for propositional variables of $\mathcal{L}$. The forcing relation $\mathcal{M}, w \Vdash_{st} A$ is defined as follows, where $[\![A]\!]_{\mathcal{M}}$ denotes the set $\{v \mid \mathcal{M}, v \Vdash_{st} A\}$, also called the *truth set* of $A$:

$$
\begin{array}{lll}
\mathcal{M}, w \Vdash_{st} p & \text{iff} & w \in V(p); \\
\mathcal{M}, w \nVdash_{st} \bot; & & \\
\mathcal{M}, w \Vdash_{st} \top; & & \\
\mathcal{M}, w \Vdash_{st} A \wedge B & \text{iff} & \mathcal{M}, w \Vdash_{st} A \text{ and } \mathcal{M}, w \Vdash_{st} B; \\
\mathcal{M}, w \Vdash_{st} A \vee B & \text{iff} & \mathcal{M}, w \Vdash_{st} A \text{ or } \mathcal{M}, w \Vdash_{st} B; \\
\mathcal{M}, w \Vdash_{st} A \to B & \text{iff} & \mathcal{M}, w \nVdash_{st} A \text{ or } \mathcal{M}, w \Vdash_{st} B; \\
\mathcal{M}, w \Vdash_{st} \Box A & \text{iff} & [\![A]\!]_{\mathcal{M}} \in \mathcal{N}(w).
\end{array}
$$

We adopt the standard definitions of validity.

**Definition 3.2.** We say that a formula $A$ is *valid* in a model $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$, written $\mathcal{M} \models A$, if $\mathcal{M}, w \Vdash A$ for all $w \in \mathcal{W}$. We say that $A$ is *valid* in a class of models $\mathcal{C}$, written $\mathcal{C} \models A$, if $\mathcal{M} \models A$ for all $\mathcal{M} \in \mathcal{C}$.

The class of all standard models characterises the basic logic **E**. For the extensions of logic **E**, we need to consider some additional closure properties of the neighbourhood function, as specified in the next definition.

**Definition 3.3** (Semantic conditions for Extensions)**.** Given a standard neighbourhood model $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$, We consider the following conditions on $\mathcal{N}$

| | |
|---|---|
| $(M)$ | If $\alpha \in \mathcal{N}(w)$ and $\alpha \subseteq \beta$, then $\beta \in \mathcal{N}(w)$. |
| $(C)$ | If $\alpha, \beta \in \mathcal{N}(w)$, then $\alpha \cap \beta \in \mathcal{N}(w)$. |
| $(N)$ | $\mathcal{W} \in \mathcal{N}(w)$. |
| $(T)$ | If $\alpha \in \mathcal{N}(w)$, then $w \in \alpha$. |
| $(P)$ | $\emptyset \notin \mathcal{N}(w)$. |
| $(D)$ | If $\alpha \in \mathcal{N}(w)$, then $\mathcal{W} \setminus \alpha \notin \mathcal{N}(w)$. |
| $(RD_n^+)$ | If $\alpha_1, ..., \alpha_n \in \mathcal{N}(w)$, then $\alpha_1 \cap ... \cap \alpha_n \neq \emptyset$. |

The properties $(M)$, $(C)$, $(N)$ are respectively called *supplementation*, *closure under intersection*, and *containing the unit*; accordingly, a standard model is supplemented, closed under intersection, or contains the unit if it satisfies the corresponding property.

In the following, for every logic **L** we denote by $\mathcal{C}^{st}_{\mathbf{L}}$ the class of standard models for **L**.

**Theorem 3.1.** Logic $\mathbf{E}^*$ is sound and complete with respect to the corresponding standard neighbourhood models, that is: $\models_{\mathcal{C}^{st}_{\mathbf{E}^*}} A$ if and only if $\vdash_{\mathbf{E}^*} A$.

A proof of this result can be found in [4] for the systems of the classical cube and their extensions with axioms $T$, $P$, $D$. The proof can be easily extended also to the rules $RD_n^+$.

A special case is given by regular logics $\mathbf{MC}^*$ (i.e., possibly lacking the necessitation axiom $N$). For these logics there exists a relational semantics which goes back to Kripke himself: in [23] he introduces relational models with so-called non-normal worlds, with the aim of characterising a family of Lewis' and Lemmon's systems in which necessitation fails or is validated only in a restricted form. Here we consider a definition from Priest [35, Section 4.2].

**Definition 3.4.** A *relational model* with *non-normal worlds* is a tuple $\mathcal{M} = \langle \mathcal{W}, \mathcal{W}^i, \mathcal{R}, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set of worlds, $\mathcal{W}^i \subseteq \mathcal{W}$ is the set of non-normal worlds, $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{W}$ is a binary relation, and $\mathcal{V} : Atm \longrightarrow \mathcal{P}(\mathcal{W})$ is a valuation function for propositional variables. The forcing relation $w \Vdash_r A$ is defined as in Definition 3.1, except for boxed formulas, for which it is defined as follows:

$$w \Vdash_r \Box A \quad \text{iff} \quad w \notin \mathcal{W}^i \text{ and for all } v \in \mathcal{W}, \text{ if } w\mathcal{R}v \text{ then } v \Vdash A.$$

Observe that in non-normal worlds all boxed formulas are falsified. Validity is defined as usual: we say that a formula is valid in a model if it is true in all worlds (no matter if they are normal or non-normal).[1] It is easy to verify that non-normal relational models validate axioms $M$ and $C$ but do not validate

[1] This is one of the *two* definitions of validity considered by Priest [35].

axiom $N$. Notice also that in case $\mathcal{W}^i$ is empty we have the standard definition of Kripke models for normal modal logics.

A semantic characterisation of some extensions of logic **MC** by means of non-normal relational models can be given by considering the usual frame properties of Kripke semantics.

**Theorem 3.2** ([12, pp. 300])**.** Every *regular* logic $\mathbf{MC}^*$ is sound and complete with respect to the corresponding relational models.

A variant of the neighbourhood semantics, called *bi-neighbourhood* semantics, was introduced in [9]. Instead of a set of neighbourhoods, worlds in bi-neighbourhood models are equipped with a set of *pairs* of neighbourhoods. The intuition is that the two components of a pair provide "positive" and "negative" support for a modal formula. As we shall see, a technical motivation to consider bi-neighhbourhood models is that they are more suitable for countermodel generation than standard ones, but they also have an interest in their own.

**Definition 3.5.** A *bi-neighbourhood model* is a triple $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set, $\mathcal{V}$ is a valuation function and $\mathcal{N}$ is a function assigning to each world $w$ a subset of $\mathcal{P}(\mathcal{W}) \times \mathcal{P}(\mathcal{W})$. The forcing relation $\mathcal{M}, w \Vdash_{bi} A$ is defined as in Definition 3.1 except for the modality, for which the clause is:

$$\mathcal{M}, w \Vdash_{bi} \Box A \quad \text{iff} \quad \text{there is } (\alpha, \beta) \in \mathcal{N}(w) \text{ s.t. } \alpha \subseteq [\![A]\!]_{\mathcal{M}} \subseteq \mathcal{W} \setminus \beta.$$

The above definition introduces the general class of bi-neighbourhood models, thus characterising the basic logic **E**. For extensions we need to consider the further conditions contained in next definition.

**Definition 3.6** (Bi-neighbourhood conditions for extensions)**.** Given a bi-neighbourhood model $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$, we consider the following conditions:

| | |
|---|---|
| (M) | If $(\alpha, \beta) \in \mathcal{N}(w)$, then $\beta = \emptyset$. |
| (N) | There is $\alpha \subseteq \mathcal{W}$ such that for all $w \in \mathcal{W}$, $(\alpha, \emptyset) \in \mathcal{N}(w)$. |
| (C) | If $(\alpha, \beta), (\gamma, \delta) \in \mathcal{N}(w)$, then $(\alpha \cap \gamma, \beta \cup \delta) \in \mathcal{N}(w)$. |
| (T) | If $(\alpha, \beta) \in \mathcal{N}(w)$, then $w \in \alpha$. |
| (P) | If $(\alpha, \beta) \in \mathcal{N}(w)$, then $\alpha \neq \emptyset$. |
| (D) | If $(\alpha, \beta), (\gamma, \delta) \in \mathcal{N}(w)$, then $\alpha \cap \gamma \neq \emptyset$ or $\beta \cap \delta \neq \emptyset$. |
| ($\mathrm{RD}_n^+$) | If $(\alpha_1, \beta_1), ..., (\alpha_n, \beta_n) \in \mathcal{N}(w)$, then $\alpha_1 \cap ... \cap \alpha_n \neq \emptyset$. |

In the following, we simply write $w \Vdash A$ and $[\![A]\!]$, omitting both the model $\mathcal{M}$ and the subscript *bi*, *st*, or *r* whenever they are clear from the context.

For every condition (X) above, we call *X-model* any bi-neighbourhood model statisfying (X). The class of bi-neighbourhood models for a given non-normal modal logic **L** is determined by the conditions corresponding to the axioms of **L**. We denote by $\mathcal{C}_{\mathbf{L}}^{bi}$ the class of bi-neighbourhood models for **L**.

We now prove that non-normal modal logics are sound and complete with respect to the corresponding classes of bi-neighbourhood models. For the systems of the classical cube, a direct proof based on the canonical model construction can be found in [9]. Here we adopt a different approach: by showing a mutual transformation between standard and bi-neighbourhood-models, completeness of non-normal modal logics with respect to standard models appears as a by-product.

First of all, given a standard model, an equivalent bi-neighbourhood model can be obtained simply by taking as pairs each neighbourhood and its complement (for the classical cube this transformation is already introduced in [9]).

**Proposition 3.3.** Let $\mathcal{M}_{st} = \langle \mathcal{W}, \mathcal{N}_{st}, \mathcal{V} \rangle$ be a standard model, and $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}_{bi}, \mathcal{V} \rangle$ be the bi-neighbourhood model defined by taking the same $\mathcal{W}$ and $\mathcal{V}$ and, for all $w \in \mathcal{W}$,

$$\mathcal{N}_{bi}(w) = \begin{cases} \{(\alpha, \mathcal{W} \setminus \alpha) \mid \alpha \in \mathcal{N}_{st}(w)\} & \text{if } \mathcal{M}_{st} \text{ is not supplemented.} \\ \{(\alpha, \emptyset) \mid \alpha \in \mathcal{N}_{st}(w)\} & \text{if } \mathcal{M}_{st} \text{ is supplemented.} \end{cases}$$

Then for every formula $A$ of $\mathcal{L}$ and every $w \in \mathcal{W}$, $\mathcal{M}_{bi}, w \Vdash A$ if and only if $\mathcal{M}_{st}, w \Vdash A$. Moreover, for every $X \in \{M,\ C,\ N,\ T,\ P,\ D,\ RD_n^+\}$, if $\mathcal{M}_{st}$ satisfies the condition corresponding to $X$ in the standard semantics, then $\mathcal{M}_{bi}$ is a bi-neighbourhood X-model.

*Proof.* The equivalence is proved by induction on $A$. The basic cases $A = p, \bot, \top$ are trivial since the evaluation $\mathcal{V}$ is the same in the two models, and the inductive cases of boolean connectives are straightforward by applying the induction hypothesis. We consider the case $A = \Box B$. If $\mathcal{M}_{st}$ is not supplemented we have: $\mathcal{M}_{bi}, w \Vdash \Box B$ iff $([\![B]\!]_{bi}, \mathcal{W} \setminus [\![B]\!]_{bi}) \in \mathcal{N}_{bi}(w)$ iff $[\![B]\!]_{bi} \in \mathcal{N}_{st}(w)$ iff (inductive hypothesis) $[\![B]\!]_{st} \in \mathcal{N}_{st}(w)$ iff $\mathcal{M}_{st}, w \Vdash \Box B$. If $\mathcal{M}_{st}$ is supplemented we have: $\mathcal{M}_{bi}, w \Vdash \Box B$ iff there is $(\alpha, \emptyset) \in \mathcal{N}_{bi}(w)$ such that $\alpha \subseteq [\![B]\!]_{bi}$ iff $\alpha \in \mathcal{N}_{st}(w)$ and (inductive hypothesis) $\alpha \subseteq [\![B]\!]_{st}$ iff (by supplementation) $[\![B]\!]_{st} \in \mathcal{N}_{st}(w)$ iff $\mathcal{M}_{st}, w \Vdash \Box B$.

Now we show that $\mathcal{M}_{st}$ satisfies the right properties. For axiom $M$ the proof is immediate by definition of $\mathcal{N}_{bi}$. For the following conditions we just consider the non-supplemented case, whereas the supplemented case is an easy simplification.

$(N)$ $(\mathcal{W}, \emptyset) \in \mathcal{N}_{bi}(w)$ because $\mathcal{W} \in \mathcal{N}_{st}(w)$.

$(C)$ If $(\alpha, \mathcal{W} \setminus \alpha), (\beta, \mathcal{W} \setminus \beta) \in \mathcal{N}_{bi}(w)$, then $\alpha, \beta \in \mathcal{N}_{st}(w)$, that implies $\alpha \cap \beta \in \mathcal{N}_{st}(w)$. Thus $(\alpha \cap \beta, \mathcal{W} \setminus (\alpha \cap \beta)) = (\alpha \cap \beta, \mathcal{W} \setminus \alpha \cup \mathcal{W} \setminus \beta) \in \mathcal{N}_{bi}(w)$.

$(T)$ If $(\alpha, \mathcal{W} \setminus \alpha) \in \mathcal{N}_{bi}(w)$, then $\alpha \in \mathcal{N}_{st}(w)$, thus $w \in \alpha$.

$(P)$ If $(\alpha, \mathcal{W} \setminus \alpha) \in \mathcal{N}_{bi}(w)$, then $\alpha \in \mathcal{N}_{st}(w)$, thus $\alpha \neq \emptyset$.

$(D)$ If $(\alpha, \mathcal{W} \setminus \alpha), (\beta, \mathcal{W} \setminus \beta) \in \mathcal{N}_{bi}(w)$, then $\alpha, \beta \in \mathcal{N}_{st}(w)$. Thus $\beta \neq \mathcal{W} \setminus \alpha$, that implies $\alpha \cap \beta \neq \emptyset$ or $\mathcal{W} \setminus \alpha \cap \mathcal{W} \setminus \beta \neq \emptyset$.

$(RD_n^+)$ If $(\alpha_1, \mathcal{W} \setminus \alpha_1), ..., (\alpha_n, \mathcal{W} \setminus \alpha_n) \in \mathcal{N}_{bi}(w)$, then $\alpha_1, ..., \alpha_n \in \mathcal{N}_{st}(w)$, thus $\alpha_1 \cap ... \cap \alpha_n \neq \emptyset$. $\Box$

Fo the opposite direction we propose two transformations: a more general one, independent of the language, and a "finer" one which is defined with respect to a set of formulas. The general transformation is new, whereas the second one is already introduced in [9] for the classical cube, where the equivalence proof is also sketched. The general transformation is as follows.

**Proposition 3.4.** Let $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}_{bi}, \mathcal{V} \rangle$ be a bi-neighbourhood model, and $\mathcal{M}_{st} = \langle \mathcal{W}, \mathcal{N}_{st}, \mathcal{V} \rangle$ be the standard model defined by taking the same $\mathcal{W}$ and $\mathcal{V}$ and, for all $w \in \mathcal{W}$,

$$\mathcal{N}_{st}(w) = \{\gamma \subseteq \mathcal{W} \mid \text{there is } (\alpha, \beta) \in \mathcal{N}_{bi}(w) \text{ such that } \alpha \subseteq \gamma \subseteq \mathcal{W} \setminus \beta\}.$$

Then for every formula $A$ of $\mathcal{L}$ and every $w \in \mathcal{W}$, $\mathcal{M}_{st}, w \Vdash A$ if and only if $\mathcal{M}_{bi}, w \Vdash A$. Moreover, for every $X \in \{M, C, N, T, P, D, RD_n^+\}$, if $\mathcal{M}_{bi}$ is a bi-neighbourhood X-model, then $\mathcal{M}_{st}$ satisfies the condition corresponding to $X$ in the standard semantics.

*Proof.* The equivalence is proved by induction on $A$. As before, we only consider the inductive step where $A \equiv \Box B$. We have: $\mathcal{M}_{st}, w \Vdash \Box B$ iff $[\![B]\!]_{st} \in \mathcal{N}_{st}(w)$ iff (i.h.) $[\![B]\!]_{bi} \in \mathcal{N}_{st}(w)$ iff there is $(\alpha, \beta) \in \mathcal{N}_{bi}(w)$ such that $\alpha \subseteq [\![B]\!]_{bi} \subseteq \mathcal{W} \setminus \beta$ iff $\mathcal{M}_{bi}, w \Vdash \Box B$.

Now we prove that $\mathcal{M}_{st}$ satisfies the right properties.

$(M)$ Let $\mathcal{M}_{bi}$ be a M-model, and assume $\gamma \in \mathcal{N}_{st}(w)$ and $\gamma \subseteq \delta$. Then there is $(\alpha, \emptyset) \in \mathcal{N}_{bi}(w)$ such that $\alpha \subseteq \gamma \subseteq \mathcal{W} \setminus \emptyset$. Thus $\alpha \subseteq \delta \subseteq \mathcal{W} \setminus \emptyset$, which implies $\delta \in \mathcal{N}_{st}(w)$.

$(N)$ Let $\mathcal{M}_{bi}$ be a N-model. Then there is $(\alpha, \emptyset) \in \mathcal{N}_{bi}(w)$. Since $\alpha \subseteq \mathcal{W} \subseteq \mathcal{W} \setminus \emptyset$, by definition $\mathcal{W} \in \mathcal{N}_{st}(w)$.

$(C)$ Let $\mathcal{M}_{bi}$ be a C-model, and assume $\gamma, \delta \in \mathcal{N}_{st}(w)$. Then there are $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathcal{N}_{bi}(w)$ such that $\alpha_1 \subseteq \gamma \subseteq \mathcal{W} \setminus \beta_1$, $\alpha_2 \subseteq \delta \subseteq \mathcal{W} \setminus \beta_2$. By condition (C), $(\alpha_1 \cap \alpha_2, \beta_1 \cup \beta_2) \in \mathcal{N}_{bi}(w)$, where $\alpha_1 \cap \alpha_2 \subseteq \gamma \cap \delta$, and $\gamma \cap \delta \subseteq \mathcal{W} \setminus \beta_1 \cap \mathcal{W} \setminus \beta_2 = \mathcal{W} \setminus \beta_1 \cup \beta_2$. Then $\gamma \cap \delta \in \mathcal{N}_{st}(w)$.

$(T)$ Let $\mathcal{M}_{bi}$ be a T-model, and assume $\gamma \in \mathcal{N}_{st}(w)$. Then there is $(\alpha, \beta) \in \mathcal{N}_{bi}(w)$ such that $\alpha \subseteq \gamma \subseteq \mathcal{W} \setminus \beta$. By condition (T), $w \in \alpha$, then $w \in \gamma$.

$(P)$ Let $\mathcal{M}_{bi}$ be a P-model, and assume towards contradiction that $\emptyset \in \mathcal{N}_{st}(w)$. Then there is $(\alpha, \beta) \in \mathcal{N}_{bi}(w)$ such that $\alpha \subseteq \emptyset \subseteq \mathcal{W} \setminus \beta$. Thus $\alpha = \emptyset$, against condition (P).

$(D)$ Let $\mathcal{M}_{bi}$ be a D-model, and assume towards contradiction that $\gamma, \mathcal{W} \setminus \gamma \in \mathcal{N}_{st}(w)$. Then there are $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathcal{N}_{bi}(w)$ such that $\alpha_1 \subseteq \gamma \subseteq \mathcal{W} \setminus \beta_1$, $\alpha_2 \subseteq \mathcal{W} \setminus \gamma \subseteq \mathcal{W} \setminus \beta_2$. Then $\alpha_1 \cap \alpha_2 = \emptyset$ and $\beta_1 \cap \beta_2 = \emptyset$, against condition (D).

$(RD_n^+)$ Let $\mathcal{M}_{bi}$ be a $\mathrm{RD}_n^+$-model, and assume $\gamma_1, ..., \gamma_n \in \mathcal{N}_{st}(w)$. Then there are $(\alpha_1, \beta_1), ..., (\alpha_n, \beta_n) \in \mathcal{N}_{bi}(w)$ such that $\alpha_i \subseteq \gamma \subseteq \mathcal{W} \setminus \beta_i$ for all $1 \leq i \leq n$. By condition ($\mathrm{RD}_n^+$), $\alpha_1 \cap ... \cap \alpha_n \neq \emptyset$. Then $\gamma_1 \cap ... \cap \gamma_n \neq \emptyset$. $\Box$

For the non-monotonic case, we provide another transformation, defined with respect to any set of formulas $\mathcal{S}$ closed under subformulas. This transformation in general produces standard models with a *smaller* neighbourhood function.

**Proposition 3.5.** Let $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}_{bi}, \mathcal{V} \rangle$ be a bi-neighbourhood model and $\mathcal{S}$ be a set of formulas of $\mathcal{L}$ closed under subformulas. We define the standard model $\mathcal{M}_{st} = \langle \mathcal{W}, \mathcal{N}_{st}, \mathcal{V} \rangle$ with the same $\mathcal{W}$ and $\mathcal{V}$ and by taking, for all $w \in \mathcal{W}$,

$$\mathcal{N}_{st}(w) = \{[\![A]\!]_{bi} \mid \Box A \in \mathcal{S} \text{ and } \mathcal{M}_{bi}, w \Vdash \Box A\}.$$

Then for every formula $A \in \mathcal{S}$ and every world $w \in \mathcal{W}$, $\mathcal{M}_{st}, w \Vdash A$ if and only if $\mathcal{M}_{bi}, w \Vdash A$. Moreover, (N) if $\Box\top \in \mathcal{S}$ and $\mathcal{M}_{bi}$ is a N-model, then $\mathcal{M}_{st}$ contains the unit; (C) if $\Box A, \Box B \in \mathcal{S}$ implies $\Box(A \wedge B) \in \mathcal{S}$ and $\mathcal{M}_{bi}$ is a C-model, then $\mathcal{M}_{st}$ is closed under intersection; (T/P/D/$\mathrm{RD}_n^+$) If $\mathcal{M}_{bi}$ is a T/P/D/$\mathrm{RD}_n^+$-model, then $\mathcal{M}_{st}$ satisfies the corresponding condition in the standard semantics.

*Proof.* The equivalence is proved by induction on $A$. The basic cases are immediate. If $A \equiv B \circ C$, where $\circ \in \{\wedge, \vee, \rightarrow\}$, the claims holds by applying the inductive hypothesis since $B, C \in \mathcal{S}$ because $\mathcal{S}$ is closed under subformulas. If $A \equiv \Box B$, then $B \in \mathcal{S}$ and, by inductive hypothesis, $[\![B]\!]_{st} = [\![B]\!]_{bi}$. Thus

$\mathcal{M}_{st}, w \Vdash \Box B$ iff $[\![B]\!]_{st} \in \mathcal{N}_{st}(w)$ iff $[\![B]\!]_{bi} \in \mathcal{N}_{st}(w)$ iff there is $\Box C \in \mathcal{S}$ such that $[\![C]\!]_{bi} = [\![B]\!]_{bi}$ and $\mathcal{M}_{st}, w \Vdash \Box C$ iff $\mathcal{M}_{st}, w \Vdash \Box B$.

(N) Let $\mathcal{M}_{bi}$ be a N-model. Then $\mathcal{M}_{bi}, w \Vdash \Box\top$. Since $\Box\top \in \mathcal{S}$, by definition $[\![\top]\!]_{bi} = \mathcal{W} \in \mathcal{N}_{st}(w)$.

(C) Assume $\alpha, \beta \in \mathcal{N}_{st}(w)$. Then there are $\Box A, \Box B \in \mathcal{S}$ such that $\alpha = [\![A]\!]_{bi}$, $\beta = [\![B]\!]_{bi}$, and $\mathcal{M}_{bi}, w \Vdash \Box A$, $\mathcal{M}_{bi}, w \Vdash \Box B$, that is $\mathcal{M}_{bi}, w \Vdash \Box A \wedge \Box B$. Since $\mathcal{M}_{bi}$ is a C-model we have $\mathcal{M}_{bi}, w \Vdash \Box(A \wedge B)$. By the properties of $\mathcal{S}$, $\Box(A \wedge B) \in \mathcal{S}$. Then by definition $[\![A \wedge B]\!]_{bi} \in \mathcal{N}_{st}(w)$, where $[\![A \wedge B]\!]_{bi} = [\![A]\!]_{bi} \cap [\![B]\!]_{bi} = \alpha \cap \beta$.

(T) Assume $\alpha \in \mathcal{N}_{st}(w)$. Then $\alpha = [\![A]\!]_{bi}$ for some $A$ such that $\Box A \in \mathcal{S}$ and $\mathcal{M}_{bi}, w \Vdash \Box A$. Since $\mathcal{M}_{bi}$ is a T-model, $\mathcal{M}_{bi}, w \Vdash A$, that is $w \in [\![A]\!]_{bi} = \alpha$.

(P) Assume by contradiction that $\emptyset \in \mathcal{N}_{st}(w)$. Then there is $\Box A \in \mathcal{S}$ such that $\mathcal{M}_{bi}, w \Vdash \Box A$ and $[\![A]\!]_{bi} = \emptyset = [\![\bot]\!]_{bi}$. Thus $\mathcal{M}_{bi}, w \Vdash \Box\bot$, against the soundness of axiom $P$ with respect to P-models.

(D) Assume $\alpha, \mathcal{W} \setminus \alpha \in \mathcal{N}_{st}(w)$. Then there are $\Box A, \Box B \in \mathcal{S}$ such that $\alpha = [\![A]\!]_{bi}$, $\mathcal{W} \setminus \alpha = [\![B]\!]_{bi}$, and $\mathcal{M}_{bi}, w \Vdash \Box A$, $\mathcal{M}_{bi}, w \Vdash \Box B$. Then $[\![A]\!]_{bi} = \mathcal{W} \setminus [\![B]\!]_{bi} = [\![\neg B]\!]_{bi}$, that is $\mathcal{M}_{bi}, w \Vdash \Box\neg B$, against the soundness of axiom $D$ with respect to D-models.

($\mathrm{RD}_n^+$) Assume $\alpha_1, ..., \alpha_n \in \mathcal{N}_{st}(w)$. Then there are $\Box A_1, ..., \Box A_n \in \mathcal{S}$ such that $\alpha_i = [\![A_i]\!]_{bi}$ and $\mathcal{M}_{bi}, w \Vdash \Box A_i$ for every $1 \leq i \leq n$, that is $\mathcal{M}_{bi}, w \Vdash \Box A_1 \wedge ... \wedge \Box A_n$. Then $\mathcal{M}_{bi} \not\models \neg(\Box A_1 \wedge ... \wedge \Box A_n)$, and since $\mathcal{M}_{bi}$ is a $\mathrm{RD}_n^+$-model, $\mathcal{M}_{bi} \not\models \neg(A_1 \wedge ... \wedge A_n)$, that is $[\![A_1]\!]_{bi} \cap ... \cap [\![A_n]\!]_{bi} = \alpha_1 \cap ... \cap \alpha_n \neq \emptyset$. $\Box$

For the monotonic case, an analogous result could be obtained by considering the *supplementation* of the neighbourhood function in the above proposition. That is we can consider

$$\mathcal{N}'_{st}(w) = \{\alpha \subseteq \mathcal{W} \mid \text{there is } \Box A \in \mathcal{S} \text{ such that } \mathcal{M}_{bi}, w \Vdash \Box A \text{ and } [\![A]\!]_{bi} \subseteq \alpha\}.$$

However, in this case the advantage in the size of the neighbourhood function with respect to the transformation in Proposition 3.4 is not as relevant as for the non-monotonic case.

From Propositions 3.3 and 3.4, standard and bi-neighbourhood semantics are equivalent, in the sense that a formula is valid in a certain class of standard models if and only if it is valid in the corresponding class of bi-neighbourhood models. Since non-normal modal logics are characterised by standard models, we obtain the following result.

**Theorem 3.6** (Characterisation)**.** Every non-normal modal logic $\mathbf{L}$ is sound and complete with respect to the corresponding class of bi-neighbourhood models, that is: for every $A \in \mathcal{L}$, $\models_{\mathcal{C}_{\mathbf{L}}^{bi}} A$ if and only if $\vdash_{\mathbf{L}} A$.

We conclude this section with a few observations. First of all, the bi-neighbourhood/neighbourhood semantics is (more) significant for the non-monotonic systems, that is lacking the axiom $M$: for the monotonic ones, the truth condition for $\Box$ in bi-neighbourhood-models boils down to the well-known $\exists\forall$-definition found in the literature (see e.g. [32]). Moreover, by the transformation presented in Proposition 3.4, elements of bi-neighbourhood pairs can be seen as *lower* and *upper* bounds of neighbourhoods of standard models.

These transformations have also an interest in proof-search: as we shall see in the following, given a failed proof in our calculi, it is possible to directly extract a countermodel in the bi-neighbourhood semantics, that can be transformed into

an equivalent standard countermodel as a later step. Furthermore, whereas in a standard model each (non-equivalent) boxed formula must be "witnessed" by a different neighbourhood, the same bi-neighbourhood pair can "witness" several boxed formulas. For this reason, bi-neighbourhood countermodels extracted from failed proofs have typically a smaller neighbourhood function than the corresponding standard models. A simple example is given by the formula

$$\Box(p \wedge q) \wedge \Box(p \wedge \neg q) \wedge \Box(\neg p \wedge q) \wedge \Box(\neg p \wedge \neg q).$$

A standard neighbourhood function to validate this formula would need to include a separate set for each of the boolean combinations of the variables $p$ and $q$, whereas a bi-neighbourhood neighbourhood function validating this formula could contain only the single tuple $(\emptyset, \emptyset)$.

# 4 Hypersequent calculi

In order to define our calculi, we extend the structure of sequents in two ways. Firstly, sequents can contain so-called *blocks of formulas* in addition to formulas of $\mathcal{L}$. Secondly, we use *hypersequents* rather than simple sequents.

**Definition 4.1.** A *block* is a structure $\langle \Sigma \rangle$, where $\Sigma$ is a finite multiset of formulas of $\mathcal{L}$. A *sequent* is a pair $\Gamma \Rightarrow \Delta$, where $\Gamma$ is a finite multiset of formulas and blocks, and $\Delta$ is a finite multiset of formulas. A *hypersequent* is a finite multiset of sequents, and is written

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n.$$

Given a hypersequent $H = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$, we call *components* of $H$ the sequents $\Gamma_i \Rightarrow \Delta_i,\ 1 \leq i \leq n$.

Intuitively, blocks can be seen as a structural connective corresponding to boxes in the antecedent of a sequent. The location in the antecedent enables a simple formulation of distribution of boxes over conjunction. Both blocks and sequents, but not hypersequents, can be interpreted as formulas of $\mathcal{L}$. The *formula interpretation* of sequents is as follows:

$$i(A_1, \ldots, A_n, \langle \Sigma_1 \rangle, \ldots, \langle \Sigma_m \rangle \Rightarrow B_1, \ldots, B_k) =$$
$$A_1 \wedge \ldots \wedge A_n \wedge \Box \bigwedge \Sigma_1 \wedge \ldots \wedge \Box \bigwedge \Sigma_m \rightarrow B_1 \vee \ldots \vee B_k.$$

By contrast, there is no formula interpretation for hypersequents in $\mathcal{L}$. The reason is that non-normal modalities are not *strong enough* to express the structural connective "|" of hypersequents: Intuitively, every component of a hypersequent corresponds to a world in a model, and all worlds of a model are potentially relevant for calculating the truth set of a formula, so we would need a global modality to express the hypersequent structure.

The semantic interpretation of sequents and hypersequents is as follows.

**Definition 4.2.** We say that a sequent $S$ is *valid* in a possible-worlds model $\mathcal{M}$ (written $\mathcal{M} \models S$) if for every world $w$ of $\mathcal{M}$, $\mathcal{M}, w \Vdash i(S)$. We say that a hypersequent $H$ is *valid* in $\mathcal{M}$ if for some component $S$ of $H$, $\mathcal{M} \models S$. Finally, we say that an inference rule $R$ is *sound* with respect to a model $\mathcal{M}$ (resp. a class of models $\mathcal{C}$) if in case all premisses of $R$ are valid in $\mathcal{M}$ (resp. $\mathcal{C}$), then the conclusion of $R$ is also valid in $\mathcal{M}$ (resp. $\mathcal{C}$).

**Propositional rules**

$$\mathsf{init}\ \frac{}{G \mid \Gamma, p \Rightarrow p, \Delta} \qquad \mathsf{L}\bot\ \frac{}{G \mid \Gamma, \bot \Rightarrow \Delta} \qquad \mathsf{R}\top\ \frac{}{G \mid \Gamma \Rightarrow \top, \Delta}$$

$$\mathsf{L}\to\ \frac{G \mid \Gamma, A \to B \Rightarrow A, \Delta \qquad G \mid \Gamma, A \to B, B \Rightarrow \Delta}{G \mid \Gamma, A \to B \Rightarrow \Delta} \quad \mathsf{R}\to\ \frac{G \mid \Gamma, A \Rightarrow B, A \to B, \Delta}{G \mid \Gamma \Rightarrow A \to B, \Delta}$$

$$\mathsf{L}\wedge\ \frac{G \mid \Gamma, A \wedge B, A, B \Rightarrow \Delta}{G \mid \Gamma, A \wedge B \Rightarrow \Delta} \quad \mathsf{R}\wedge\ \frac{G \mid \Gamma \Rightarrow A, A \wedge B, \Delta \qquad G \mid \Gamma \Rightarrow B, A \wedge B, \Delta}{G \mid \Gamma \Rightarrow A \wedge B, \Delta}$$

$$\mathsf{L}\vee\ \frac{G \mid \Gamma, A \vee B, A \Rightarrow \Delta \qquad G \mid \Gamma, A \vee B, B \Rightarrow \Delta}{G \mid \Gamma, A \vee B \Rightarrow \Delta} \quad \mathsf{R}\vee\ \frac{G \mid \Gamma \Rightarrow A, B, A \vee B, \Delta}{G \mid \Gamma \Rightarrow A \vee B, \Delta}$$

**Modal rules for the classical cube**

$$\mathsf{L}\Box\ \frac{G \mid \Gamma, \Box A, \langle A \rangle \Rightarrow \Delta}{G \mid \Gamma, \Box A \Rightarrow \Delta} \qquad \mathsf{R}\Box\mathsf{m}\ \frac{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid \Sigma \Rightarrow B}{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta}$$

$$\mathsf{R}\Box\ \frac{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid \Sigma \Rightarrow B \qquad \{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid B \Rightarrow A\}_{A \in \Sigma}}{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta}$$

$$\mathsf{N}\ \frac{G \mid \Gamma, \langle \top \rangle \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta} \qquad \mathsf{C}\ \frac{G \mid \Gamma, \langle \Sigma \rangle, \langle \Pi \rangle, \langle \Sigma, \Pi \rangle \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma \rangle, \langle \Pi \rangle \Rightarrow \Delta}$$

**Modal rules for extensions**

$$\mathsf{T}\ \frac{G \mid \Gamma, \langle \Sigma \rangle, \Sigma \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Delta} \qquad \mathsf{P}\ \frac{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Delta \mid \Sigma \Rightarrow}{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Delta}$$

$$\mathsf{D}_n^+\ \frac{G \mid \Gamma, \langle \Sigma_1 \rangle, \ldots, \langle \Sigma_n \rangle \Rightarrow \Delta \mid \Sigma_1, \ldots, \Sigma_n \Rightarrow}{G \mid \Gamma, \langle \Sigma_1 \rangle, \ldots, \langle \Sigma_n \rangle \Rightarrow \Delta}$$

$$\mathsf{D}_2\ \frac{G \mid \Gamma, \langle \Sigma \rangle, \langle \Pi \rangle \Rightarrow \Delta \mid \Sigma, \Pi \Rightarrow \qquad \{G \mid \Gamma, \langle \Sigma \rangle, \langle \Pi \rangle \Rightarrow \Delta \mid \Rightarrow A, B\}_{A \in \Sigma, B \in \Pi}}{G \mid \Gamma, \langle \Sigma \rangle, \langle \Pi \rangle \Rightarrow \Delta}$$

$$\mathsf{D}_1\ \frac{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Delta \mid \Sigma \Rightarrow \qquad \{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Delta \mid \Rightarrow A\}_{A \in \Sigma}}{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Delta}$$

**Figure 4:** Rules of the hypersequent calculi $\mathbf{H.E}^*$.

| | |
|---|---|
| $\mathbf{H.E} := \{$propositional rules, $\mathsf{L}\Box$, $\mathsf{R}\Box\}$. | $\mathbf{H.M} := \{$propositional rules, $\mathsf{L}\Box$, $\mathsf{R}\Box\mathsf{m}\}$. |
| $\mathbf{H.EN}^* := \mathbf{H.E}^* \cup \{\mathsf{N}\}$. | $\mathbf{H.MN}^* := \mathbf{H.M}^* \cup \{\mathsf{N}\}$. |
| $\mathbf{H.EC}^* := \mathbf{H.E}^* \cup \{\mathsf{C}\}$. | $\mathbf{H.MC}^* := \mathbf{H.M}^* \cup \{\mathsf{C}\}$. |
| $\mathbf{H.ET}^* := \mathbf{H.E}^* \cup \{\mathsf{T}\}$. | $\mathbf{H.MT}^* := \mathbf{H.M}^* \cup \{\mathsf{T}\}$. |
| $\mathbf{H.EP}^* := \mathbf{H.E}^* \cup \{\mathsf{P}\}$. | $\mathbf{H.MP}^* := \mathbf{H.M}^* \cup \{\mathsf{P}\}$. |
| $\mathbf{H.ED}^* := \mathbf{H.E}^* \cup \{\mathsf{D}_1, \mathsf{D}_2\}$. | $\mathbf{H.MD}^* := \mathbf{H.M}^* \cup \{\mathsf{D}_1^+, \mathsf{D}_2^+\}$. |
| $\mathbf{H.ED_n^+}^* := \mathbf{H.E}^* \cup \{\mathsf{D}_i^+ \mid 1 \leq i \leq n\}$. | $\mathbf{H.MD_n^+}^* := \mathbf{H.M}^* \cup \{\mathsf{D}_i^+ \mid 1 \leq i \leq n\}$. |

**Table 1:** Hypersequent calculi $\mathbf{H.E}^*$.

For every logic $\mathbf{E}^*$, the corresponding hypersequent calculus $\mathbf{H.E}^*$ is defined by a subset of the rules in Figure 4, as summarised in Table 1.

The rules are given in their cumulative, or *Kleene'd*, versions, i.e., the principal formulas or blocks are copied to the premiss(es). The propositional rules are just the hypersequent versions of *Kleene'd* rules of sequent calculi.

As mentioned in the introduction, the hypersequent structure *is not needed* to obtain a sound and complete calculus for the logics under investigation. Moreover, it can be checked that whenever a hypersequent $H = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ is derivable, then there is some component $\Gamma_i \Rightarrow \Delta_i$ which is derivable.

The choice of both the hypersequent structure and also of cumulative rules is motivated by the possibility of *directly* obtaining countermodels of non-valid formulas, that is, to define a countermodel by relying only on the information provided by a saturated branch of a failed proof, without the need of any additional computation. In particular, the hypersequent structure allows us to make all rules invertible. In this respect, observe that backward applications of rules $\mathsf{R}\Box$, $\mathsf{R}\Box\mathsf{m}$, $\mathsf{P}$, $\mathsf{D}_1$, $\mathsf{D}_2$, and $\mathsf{D}_n^+$ create new components, but the principal component in the conclusion is kept in the premiss in order keep the possibility of potential alternative rule applications.

Similarly to propositional connectives, boxed formulas are handled by separate left and right rules. Observe that rule $\mathsf{R}\Box$ has multiple premisses, but the number of the premisses is fixed by the cardinality of the principal block $\langle\Sigma\rangle$. The rule $\mathsf{R}\Box\mathsf{m}$ is a right rule for $\Box$ which replaces $\mathsf{R}\Box$ in the definition of monotonic calculi. Apart from the distinction between monotonic and non-monotonic calculi, the calculi are modular; in particular, extensions of $\mathbf{H.E}$ and $\mathbf{H.M}$ do not require to modify the basic rules for $\Box$, being defined by simply adding the rules corresponding to the additional axioms.

Every axiom has a corresponding rule, with the only exceptions of axiom $D$ and rules $RD_n^+$: axiom $D$ needs both $\mathsf{D}_1$ and $\mathsf{D}_2$ in the non-monotonic case, whereas it needs $\mathsf{D}_1^+$ and $\mathsf{D}_2^+$ in the monotonic case. Moreover, the rules $RD_n^+$ need $\mathsf{D}_m^+$ for every $1 \leq m \leq n$. These requirements makes contraction admissible (see Proposition 4.3 and Section 4.1), and we could forego them by instead adopting explicit contraction rules. Finally, as we shall see in the countermodel extraction (see Section 6), the rule $\mathsf{D}_1$ is the syntactic counterpart of the property $(\emptyset, \emptyset) \notin \mathcal{N}(w)$, which is satified by every bi-neighbourhood D-model.

Blocks have a central role in all modal rules. Modal rules essentially state how to handle blocks. Notice that the only rule which expands blocks is $\mathsf{C}$, thus in absence of this rule the blocks occurring in a proof for a single formula contain only one formula. The possibility of collecting formulas by means of blocks allows us to avoid rules with $n$ principal boxed formulas, as are common

in standard sequent calculi (compare [27]). As we shall see, blocks also allow for an easy computation of the bi-neighbourhood function for the definition of countermodels.

Derivations of modal axioms and rules are displayed in Figure 5. Note that the simulations of the rules make use of the external weakening rule $\mathsf{Ewk}$, which is shown to be admissible in Prop. 4.3. In the derivations we further implicitly make use of the following lemma, which states that initial hypersequents can be generalised to arbitrary formulas.

**Proposition 4.1.** $G \mid \Gamma, A \Rightarrow A, \Delta$ is derivable in $\mathbf{H.E}^*$ for every $A, \Gamma, \Delta, G$.

*Proof.* By structural induction on $A$. If $A = p, \bot, \top$, then $G \mid \Gamma, A \Rightarrow A, \Delta$ is an initial hypersequent, whence it is derivable. If $A = B \wedge C$ we consider the following derivation

$$\dfrac{\dfrac{G \mid \Gamma, B \wedge C, B, C \Rightarrow B, B \wedge C, \Delta \qquad G \mid \Gamma, B \wedge C, B, C \Rightarrow C, B \wedge C, \Delta}{G \mid \Gamma, B \wedge C, B, C \Rightarrow B \wedge C, \Delta}\,\mathsf{R}\wedge}{G \mid \Gamma, B \wedge C \Rightarrow B \wedge C, \Delta}\,\mathsf{L}\wedge$$

where the premisses are derivable by inductive hypothesis. The cases $A = B \vee C$ or $A = B \wedge C$ are analogous. If $A = \Box B$ we consider the following derivation

$$\dfrac{\dfrac{G \mid \Gamma, \Box B, \langle B \rangle \Rightarrow \Box B, \Delta \mid B \Rightarrow B \qquad G \mid \Gamma, \Box B, \langle B \rangle \Rightarrow \Box B, \Delta \mid B \Rightarrow B}{G \mid \Gamma, \Box B, \langle B \rangle \Rightarrow \Box B, \Delta}\,\mathsf{R}\Box}{G \mid \Gamma, \Box B \Rightarrow \Box B, \Delta}\,\mathsf{L}\Box$$

where the premisses are derivable by inductive hypothesis. $\Box$

The hypersequent calculi are sound with respect to the corresponding bi-neighbourhood models.

**Theorem 4.2** (Soundness)**.** If $H$ is derivable in $\mathbf{H.EX}^*$, then it is valid in all X-models.

*Proof.* The initial hypersequents are clearly valid. We show that all rules are sound with respect to the corresponding bi-neighbourhood models. Since the proof is standard for propositional rules, we just consider the modal rules.

($\mathsf{L}\Box$) Assume $\mathcal{M} \models G \mid \Gamma, \Box A, \langle A \rangle \Rightarrow \Delta$. Then $\mathcal{M} \models G$, or $\mathcal{M} \models \Gamma, \Box A, \langle A \rangle \Rightarrow \Delta$. In the first case we are done. In the second case, $\mathcal{M} \models i(\Gamma, \Box A, \langle A \rangle \Rightarrow \Delta) = i(\Gamma, \Box A, \Box A \Rightarrow \Delta)$, which is equivalent to $i(\Gamma, \Box A \Rightarrow \Delta)$.

($\mathsf{R}\Box$) Assume $\mathcal{M} \models G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid \Sigma \Rightarrow B$ and $\mathcal{M} \models G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid B \Rightarrow A$ for all $A \in \Sigma$. Then (i) $\mathcal{M} \models G$, or (ii) $\mathcal{M} \models \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta$, or (iii) $\mathcal{M} \models \Sigma \Rightarrow B$ and $\mathcal{M} \models B \Rightarrow A$ for all $A \in \Sigma$. If (i) or (ii) we are done. If (iii), then $\mathcal{M} \models \bigwedge \Sigma \to B$ and $\mathcal{M} \models B \to A$ for all $A \in \Sigma$, that is $\mathcal{M} \models \bigwedge \Sigma \leftrightarrow B$. Since $RE$ is valid, $\mathcal{M} \models \Box \bigwedge \Sigma \to \Box B = i(\langle \Sigma \rangle \Rightarrow \Box B)$. Thus $\mathcal{M} \models i(\Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta)$.

($\mathsf{R}\Box\mathsf{m}$) Analogous to $\mathsf{R}\Box$, by considering that in M-models $\mathcal{M} \models \bigwedge \Sigma \to B$ implies $\mathcal{M} \models \Box \bigwedge \Sigma \to \Box B$.

($\mathsf{N}$) Suppose $\mathcal{M}$ is a N-model and assume $\mathcal{M} \models G \mid \Gamma, \langle \top \rangle \Rightarrow \Delta$. Then $\mathcal{M} \models G$, or $\mathcal{M} \models \Gamma, \langle \top \rangle \Rightarrow \Delta$. In the first case we are done. In the second case, $\mathcal{M} \models i(\Gamma, \langle \top \rangle \Rightarrow \Delta)$, which is equivalent to $\Box \top \to i(\Gamma \Rightarrow \Delta)$. Since $\Box \top$ is valid in $\mathcal{M}$, $\mathcal{M} \models \Gamma \Rightarrow \Delta$.

$$(RE)\quad \dfrac{\dfrac{\dfrac{A \Rightarrow B}{\Box A, \langle A\rangle \Rightarrow \Box B \mid A \Rightarrow B}\,\mathsf{Ewk} \qquad \dfrac{B \Rightarrow A}{\Box A, \langle A\rangle \Rightarrow \Box B \mid B \Rightarrow A}\,\mathsf{Ewk}}{\Box A, \langle A\rangle \Rightarrow \Box B}\,\mathsf{R}\Box}{\Box A \Rightarrow \Box B}\,\mathsf{L}\Box$$

$$(M)\quad \dfrac{\dfrac{\dfrac{\Box(A\wedge B), \langle A\wedge B\rangle \Rightarrow \Box A \mid A\wedge B, A, B \Rightarrow A}{\Box(A\wedge B), \langle A\wedge B\rangle \Rightarrow \Box A \mid A\wedge B \Rightarrow A}\,\mathsf{L}\wedge}{\Box(A\wedge B), \langle A\wedge B\rangle \Rightarrow \Box A}\,\mathsf{R}\Box\mathsf{m}}{\Box(A\wedge B) \Rightarrow \Box A}\,\mathsf{L}\Box$$

$$(N)\quad \dfrac{\dfrac{\langle\top\rangle \Rightarrow \Box\top \mid \top \Rightarrow \top \qquad \langle\top\rangle \Rightarrow \Box\top \mid \top \Rightarrow \top}{\langle\top\rangle \Rightarrow \Box\top}\,\mathsf{R}\Box}{\Rightarrow \Box\top}\,\mathsf{N}$$

$$(C)\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\ldots, \langle A, B\rangle \Rightarrow \Box(A\wedge B) \mid A, B \Rightarrow A\wedge B \qquad \ldots \mid A\wedge B \Rightarrow A \qquad \ldots \mid A\wedge B \Rightarrow B}{\Box A\wedge\Box B, \Box A, \Box B, \langle A\rangle, \langle B\rangle, \langle A, B\rangle \Rightarrow \Box(A\wedge B)}\,\mathsf{R}\Box}{\Box A\wedge\Box B, \Box A, \Box B, \langle A\rangle, \langle B\rangle \Rightarrow \Box(A\wedge B)}\,\mathsf{C}}{\Box A\wedge\Box B, \Box A, \Box B, \langle A\rangle \Rightarrow \Box(A\wedge B)}\,\mathsf{L}\Box}{\Box A\wedge\Box B, \Box A, \Box B \Rightarrow \Box(A\wedge B)}\,\mathsf{L}\Box}{\Box A\wedge\Box B \Rightarrow \Box(A\wedge B)}\,\mathsf{L}\wedge$$

$$(T)\quad \dfrac{\dfrac{\Box A, \langle A\rangle, A \Rightarrow A}{\Box A, \langle A\rangle \Rightarrow A}\,\mathsf{T}}{\Box A \Rightarrow A}\,\mathsf{L}\Box \qquad\qquad (P)\quad \dfrac{\dfrac{\Box\bot, \langle\bot\rangle \Rightarrow \mid \bot \Rightarrow}{\Box\bot, \langle\bot\rangle \Rightarrow}\,\mathsf{P}}{\Box\bot \Rightarrow}\,\mathsf{L}\Box$$

$$(D)\quad \dfrac{\dfrac{\dfrac{\dfrac{\mathsf{L}\neg\,\dfrac{\Box A\wedge\Box\neg A, \Box A, \Box\neg A, \langle A\rangle, \langle\neg A\rangle \Rightarrow \mid A \Rightarrow A}{\Box A\wedge\Box\neg A, \Box A, \Box\neg A, \langle A\rangle, \langle\neg A\rangle \Rightarrow \mid A, \neg A \Rightarrow} \qquad \dfrac{\Box A\wedge\Box\neg A, \Box A, \Box\neg A, \langle A\rangle, \langle\neg A\rangle \Rightarrow \mid A \Rightarrow A}{\Box A\wedge\Box\neg A, \Box A, \Box\neg A, \langle A\rangle, \langle\neg A\rangle \Rightarrow \mid \Rightarrow A, \neg A}\,\mathsf{R}\neg}{\Box A\wedge\Box\neg A, \Box A, \Box\neg A, \langle A\rangle, \langle\neg A\rangle \Rightarrow}\,\mathsf{D}_2}{\Box A\wedge\Box\neg A, \Box A, \Box\neg A, \langle A\rangle \Rightarrow}\,\mathsf{L}\Box}{\Box A\wedge\Box\neg A, \Box A, \Box\neg A \Rightarrow}\,\mathsf{L}\Box}{\Box A\wedge\Box\neg A \Rightarrow}\,\mathsf{L}\wedge$$

$$(RD_n^+)\quad \dfrac{\dfrac{\dfrac{\dfrac{A_1, \ldots, A_n \Rightarrow}{\Box A_1\wedge\ldots\wedge\Box A_n, \Box A_1, \ldots, \Box A_n, \langle A_1\rangle, \ldots, \langle A_n\rangle \Rightarrow \mid A_1, \ldots, A_n \Rightarrow}\,\mathsf{Ewk}}{\Box A_1\wedge\ldots\wedge\Box A_n, \Box A_1, \ldots, \Box A_n, \langle A_1\rangle, \ldots, \langle A_n\rangle \Rightarrow}\,\mathsf{D}_n^+}{\Box A_1\wedge\ldots\wedge\Box A_n, \Box A_1, \ldots, \Box A_n \Rightarrow}\,\mathsf{L}\wedge\times n}{\Box A_1\wedge\ldots\wedge\Box A_n \Rightarrow}\,\mathsf{L}\wedge\times n$$

**Figure 5:** Derivations of modal axioms and rules.

(C) Suppose $\mathcal{M}$ is a C-model and assume $\mathcal{M} \models G \mid \Gamma, \langle\Sigma\rangle, \langle\Pi\rangle, \langle\Sigma, \Pi\rangle \Rightarrow \Delta$. Then $\mathcal{M} \models G$ or $\mathcal{M} \models \Gamma, \langle\Sigma\rangle, \langle\Pi\rangle, \langle\Sigma, \Pi\rangle \Rightarrow \Delta$. In the first case we are done. In the second case, $\mathcal{M} \models i(\Gamma, \langle\Sigma\rangle, \langle\Pi\rangle, \langle\Sigma, \Pi\rangle \Rightarrow \Delta) = i(\Gamma, \Box \bigwedge \Sigma, \Box \bigwedge \Pi, \Box(\bigwedge \Sigma \wedge \bigwedge \Pi) \Rightarrow \Delta)$. This is equivalent to $\Box \bigwedge \Sigma \wedge \Box \bigwedge \Pi \wedge \Box(\bigwedge \Sigma \wedge \bigwedge \Pi) \rightarrow i(\Gamma \Rightarrow \Delta)$, and since axiom $C$ is valid in $\mathcal{M}$, this is equivalent to $\Box \bigwedge \Sigma \wedge \Box \bigwedge \Pi \rightarrow i(\Gamma \Rightarrow \Delta)$. Thus $\mathcal{M} \models i(\Gamma, \langle\Sigma\rangle, \langle\Pi\rangle \Rightarrow \Delta)$.

(T) Suppose $\mathcal{M}$ is a T-model and assume $\mathcal{M} \models G \mid \Gamma, \langle\Sigma\rangle, \Sigma \Rightarrow \Delta$. Then $\mathcal{M} \models G$ or $\mathcal{M} \models \Gamma, \langle\Sigma\rangle, \Sigma \Rightarrow \Delta$. In the first case we are done. In the second case, $\mathcal{M} \models i(\Gamma, \langle\Sigma\rangle, \Sigma \Rightarrow \Delta) = \Box \bigwedge \Sigma \wedge \bigwedge \Sigma \rightarrow i(\Gamma \Rightarrow \Delta)$. Since axiom $T$ is valid in $\mathcal{M}$, this is equivalent to $\Box \bigwedge \Sigma \rightarrow i(\Gamma \Rightarrow \Delta)$. Then $\mathcal{M} \models \Gamma, \langle\Sigma\rangle \Rightarrow \Delta$.

(P) Suppose $\mathcal{M}$ is a P-model and assume $\mathcal{M} \models G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Delta \mid \Sigma \Rightarrow$. Then (i) $\mathcal{M} \models G$, or (ii) $\mathcal{M} \models \Gamma, \langle\Sigma\rangle \Rightarrow \Delta$, or (iii) $\mathcal{M} \models \Sigma \Rightarrow$. If (i) or (ii) we are done. If (iii), then $\mathcal{M} \models \bigwedge \Sigma \rightarrow \bot$. and by the validity of axiom $P$, $\mathcal{M} \models \Box \bigwedge \Sigma \rightarrow \bot = i(\langle\Sigma\rangle \Rightarrow)$. Then $\mathcal{M} \models \Gamma, \langle\Sigma\rangle \Rightarrow \Delta$.

($\mathsf{D}_n^+$) Suppose $\mathcal{M}$ is a $\mathrm{RD}_n^+$-model and assume $\mathcal{M} \models G \mid \Gamma, \langle\Sigma_1\rangle, \ldots, \langle\Sigma_n\rangle \Rightarrow \Delta \mid \Sigma_1, \ldots, \Sigma_n \Rightarrow$. Then (i) $\mathcal{M} \models G$, or (ii) $\mathcal{M} \models \Gamma, \langle\Sigma_1\rangle, \ldots, \langle\Sigma_n\rangle \Rightarrow \Delta$, or (iii) $\mathcal{M} \models \Sigma_1, \ldots, \Sigma_n \Rightarrow$. If (i) or (ii) we are done. If (iii), then $\mathcal{M} \models \neg(\bigwedge \Sigma_1 \wedge \ldots \wedge \bigwedge \Sigma_n)$. And by the soundness of rule $RD_n^+$, $\mathcal{M} \models \neg(\Box \bigwedge \Sigma_1 \wedge \ldots \wedge \Box \bigwedge \Sigma_n) = i(\langle\Sigma_1\rangle, \ldots, \langle\Sigma_n\rangle \Rightarrow)$. Then $\mathcal{M} \models \Gamma, \langle\Sigma_1\rangle, \ldots, \langle\Sigma_n\rangle \Rightarrow \Delta$.

($\mathsf{D}_2$) Suppose $\mathcal{M}$ is a D-model and assume $\mathcal{M} \models G \mid \Gamma, \langle\Sigma\rangle, \langle\Pi\rangle \Rightarrow \Delta \mid \Sigma, \Pi \Rightarrow$, and $\mathcal{M} \models G \mid \Gamma, \langle\Sigma\rangle, \langle\Pi\rangle \Rightarrow \Delta \mid \Rightarrow A, B$ for all $A \in \Sigma, B \in \Pi$. Then (i) $\mathcal{M} \models G$, or (ii) $\mathcal{M} \models \Gamma, \langle\Sigma\rangle, \langle\Pi\rangle \Rightarrow \Delta$, or (iii) $\mathcal{M} \models \Sigma, \Pi \Rightarrow$ and $\mathcal{M} \models \Rightarrow A, B$ for all $A \in \Sigma, B \in \Pi$. If (i) or (ii) we are done. If (iii), then $\mathcal{M} \models \bigwedge \Sigma \wedge \bigwedge \Pi \rightarrow \bot$ and $\mathcal{M} \models A \vee B$ for all $A \in \Sigma, B \in \Pi$. Thus $\mathcal{M} \models \bigwedge \Sigma \leftrightarrow \neg \bigwedge \Pi$. By the soundness of axiom $D$, $\mathcal{M} \models \Box \bigwedge \Sigma \wedge \Box \bigwedge \Pi \rightarrow \bot = i(\langle\Sigma\rangle, \langle\Pi\rangle \Rightarrow)$. Then $\mathcal{M} \models \Gamma, \langle\Sigma\rangle, \langle\Pi\rangle \Rightarrow \Delta$.

($\mathsf{D}_1$) Assume $\mathcal{M} \models G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Delta \mid \Sigma \Rightarrow$, and $\mathcal{M} \models G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Delta \mid \Rightarrow A$ for all $A \in \Sigma$. Then (i) $\mathcal{M} \models G$, or (ii) $\mathcal{M} \models \Gamma, \langle\Sigma\rangle \Rightarrow \Delta$, or (iii) $\mathcal{M} \models \Sigma \Rightarrow$ and $\mathcal{M} \models \Rightarrow A$ for all $A \in \Sigma$. If (i) or (ii) we are done. If (iii), then $\mathcal{M} \models \bigwedge \Sigma \rightarrow \bot$ and $\mathcal{M} \models \bigwedge \Sigma$, which is impossible. Then (i) or (ii) holds. $\Box$

## 4.1 Structural properties and syntactic completeness

We now investigate the structural properties of our calculi. We first show that weakening and contraction are height-preserving admissible, both in their internal and in their external variants, and that all rules are invertible. We then prove that the cut rule is admissible. This not only states the *analyticity* of the presented systems [5], but also the independence of the completeness result with respect to any particular semantics. In this way, we obtain a direct completeness proof with respect to the corresponding axiomatisations.

**Definition 4.3.** The *weight* $wg$ of a formula is recursively defined as $wg(\bot) = wg(\top) = wg(p) = 0$; for $\circ \in \{\wedge, \vee, \rightarrow\}$, $wg(A \circ B) = wg(A) + wg(B) + 1$; $wg(\langle A_1, \ldots, A_n\rangle) = max\{wg(A_1), \ldots, wg(A_n)\} + 1$; $wg(\Box A) = wg(A) + 2$.

The *height* of a derivation is the greatest number of successive applications of rules in it, where axioms have height 0. A property is *height-preserving* if the height of derivations is an invariant.

**Proposition 4.3.** The following structural rules are height-preserving admissible in $\mathbf{H.E}^*$, where $\phi$ is any formula $A$ or block $\langle\Sigma\rangle$. Moreover, all rules in $\mathbf{H.E}^*$ are height-preserving invertible.

$$\mathsf{Lwk}\ \frac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma, \phi \Rightarrow \Delta} \qquad \mathsf{Rwk}\ \frac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow A, \Delta} \qquad \mathsf{Ewk}\ \frac{G}{G \mid \Gamma \Rightarrow \Delta}$$

$$\mathsf{Lctr}\ \frac{G \mid \Gamma, \phi, \phi \Rightarrow \Delta}{G \mid \Gamma, \phi \Rightarrow \Delta} \qquad \mathsf{Rctr}\ \frac{G \mid \Gamma \Rightarrow A, A, \Delta}{G \mid \Gamma \Rightarrow A, \Delta} \qquad \mathsf{Ectr}\ \frac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}$$

$$\mathsf{Sctr}\ \frac{G \mid \Gamma, \langle \Theta, A, A \rangle \Rightarrow \Delta}{G \mid \Gamma, \langle \Theta, A \rangle \Rightarrow \Delta} \qquad \mathsf{Smgl}\ \frac{G \mid \Gamma, \langle \Theta, A \rangle \Rightarrow \Delta}{G \mid \Gamma, \langle \Theta, A, A \rangle \Rightarrow \Delta}$$

*Proof.* For each rule $R$, we prove that if the premise is derivable with height at most $n$, then the conclusion is also derivable with height at most $n$. The proof is by induction on the height of the derivation of the premise, highlighting that rules Lctr and Rctr are simultaneously proved admissible by mutual induction. Moreover, admissibility of Sctr and Smgl rely on height-preserving admissibility of contraction and weakening on formulae outside blocks, respectively.

We will illustrate with the following cases

- Case Ectr + R□m. Suppose that

$$\frac{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid \overset{\pi}{\Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid \Sigma \Rightarrow B}}{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta}\ \mathsf{R\Box m}$$

  where the height of $\pi$ is at most $n$. By induction hypothesis, there is a proof $\pi'$, with height at most $n$, of $G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid \Sigma \Rightarrow B$. Hence

$$\frac{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \overset{\pi'}{\Box B, \Delta \mid \Sigma \Rightarrow B}}{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta}\ \mathsf{R\Box m}$$

- Case Lctr + $\mathsf{D}_2$. Consider the derivation

$$\frac{G \mid \Gamma, \langle \Sigma \rangle, \langle \Sigma \rangle \overset{\pi_1}{\Rightarrow} \Delta \mid \Sigma, \Sigma \Rightarrow \quad \{G \mid \Gamma, \langle \Sigma \rangle, \langle \Sigma \rangle \overset{\pi_2^{A,B}}{\Rightarrow} \Delta \mid \Rightarrow A, B\}_{A,B \in \Sigma}}{G \mid \Gamma, \langle \Sigma \rangle, \langle \Sigma \rangle \Rightarrow \Delta}\ \mathsf{D}_2$$

  where the heights of $\pi_1$ and $\pi_2^{A,B}$ are at most $n$. Observe that, in particular, there are proofs $\pi_2^{A,A}$ of $\{G \mid \Gamma, \langle \Sigma \rangle, \langle \Sigma \rangle \Rightarrow \Delta \mid \Rightarrow A, A\}_{A \in \Sigma}$. By induction hypothesis, there are proofs $\pi_1', \pi_2^A$, with height at most $n$, of $G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Delta \mid \Sigma \Rightarrow$ and $\{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Delta \mid \Rightarrow A\}_{A \in \Sigma}$, respectively. Hence

$$\frac{G \mid \Gamma, \langle \Sigma \rangle \overset{\pi_1}{\Rightarrow} \Delta \mid \Sigma \Rightarrow \quad \{G \mid \Gamma, \langle \Sigma \rangle \overset{\pi_2^A}{\Rightarrow} \Delta \mid \Rightarrow A\}_{A \in \Sigma}}{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Delta}\ \mathsf{D}_1$$

Finally, note that since all rules are cumulative, height-preserving invertibility of all rules is an immediate consequence of height-preserving admissibility of weakening. For instance, invertibility of rule R□m is proved as follows

$$\frac{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta}{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid \Sigma \Rightarrow B}\ \mathsf{Ewk}$$

□

We note that due to the fact that the R□ rule isolates single formulae from block in its right premiss, in the non-monotonic case the full-blown weakening inside blocks is *not* admissible. However, the weaker rule of *mingle* inside blocks Smgl is.

The proof of admissibility of cut is more intricate and deserves more attention. In the hypersequent framework, the cut rule is formulated as follows

$$\mathsf{cut}\ \frac{G \mid \Gamma \Rightarrow A, \Delta \qquad G \mid \Gamma, A \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}$$

The admissibility of cut is proved simultaneously with the admissibility of the following rule sub, which states that a formula $A$ inside one or more blocks can be replaced by any equivalent multiset of formulas $\Sigma$

$$\mathsf{sub}\ \frac{G \mid \Sigma \Rightarrow A \qquad \{G \mid A \Rightarrow B\}_{B \in \Sigma} \qquad G \mid \Gamma, \langle A^{n_1}, \Pi_1 \rangle, \ldots, \langle A^{n_k}, \Pi_k \rangle \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma^{n_1}, \Pi_1 \rangle, \ldots, \langle \Sigma^{n_k}, \Pi_k \rangle \Rightarrow \Delta}$$

where $A^{n_i}$ (resp. $\Sigma^{n_i}$) is a compact way to denote $n_i$ occurrences of $A$ (resp. $\Sigma$). In the monotonic case we need to consider, instead of sub, the rule

$$\mathsf{sub_M}\ \frac{G \mid \Sigma \Rightarrow A \qquad G \mid \Gamma, \langle A^{n_1}, \Pi_1 \rangle, \ldots, \langle A^{n_k}, \Pi_k \rangle \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma^{n_1}, \Pi_1 \rangle, \ldots, \langle \Sigma^{n_k}, \Pi_k \rangle \Rightarrow \Delta}$$

**Theorem 4.4.** If $\mathbf{H.E}^*$ is non-monotonic, then the rules cut and sub are admissible in $\mathbf{H.E}^*$; otherwise cut and $\mathsf{sub_M}$ are admissible in $\mathbf{H.E}^*$.

*Proof.* We prove that cut and sub are admissible in non-monotonic $\mathbf{H.E}^*$; the proof in the monotonic case is analogous. Recall that, for an application of cut, the *cut formula* is the formula which is deleted by that application, while the *cut height* is the sum of the heights of the derivations of the premisses of cut. The theorem is a consequence of the following claims, where $Cut(c, h)$ means that all applications of cut of height $h$ on a cut formula of weight $c$ are admissible, and $Sub(c)$ means that all applications of sub where the principal formula $A$ has weight $c$ are admissible (for any $\Sigma, \Pi_1, \ldots, \Pi_k$)

- **(A)** $\forall c.Cut(c, 0)$.
- **(B)** $\forall h.Cut(0, h)$.
- **(C)** $\forall c.(\forall h.Cut(c, h) \rightarrow Sub(c))$.
- **(D)** $\forall c.\forall h.((\forall c' < c.(Sub(c') \wedge \forall h'.Cut(c', h')) \wedge \forall h'' < h.Cut(c, h'')) \rightarrow Cut(c, h))$.

**(A)** If the cut height is 0, then cut is applied to initial hypersequents $G \mid \Gamma \Rightarrow A, \Delta$ and $G \mid \Gamma, A \Rightarrow \Delta$. We show that the conclusion of cut $G \mid \Gamma \Rightarrow \Delta$ is an inital hypersequent, whence it is derivable without cut. If $G$ is an inital hypersequent we are done. Otherwise $\Gamma \Rightarrow A, \Delta$ and $\Gamma, A \Rightarrow \Delta$ are initial sequents. For the first sequent there are three possibilities: (i) $\Gamma \Rightarrow \Delta$ is an initial sequent, or (ii) $A = \top$, or (iii) $A = p$ and $\Gamma = \Gamma', p$. If (ii), then the second sequent is $\Gamma, \top \Rightarrow \Delta$, which implies that $\Gamma \Rightarrow \Delta$ is an initial sequent. If

(iii), then the second sequent is $\Gamma', p, p \Rightarrow \Delta$. Then $\Gamma' \Rightarrow \Delta$ is an initial sequent, or $\Delta = p, \Delta'$, which implies that $\Gamma', p \Rightarrow p, \Delta' = \Gamma \Rightarrow \Delta$ is an initial sequent.

**(B)** If the cut formula has weight 0, then it is $\bot$, $\top$, or a propositional variable $p$. For all three possibilities the proof is by complete induction on $h$. The basic case $h = 0$ is a particular case of **(A)**. For the inductive step, we distinguish three cases.

(i) The cut formula $\bot$, $\top$, or $p$ is not principal in the last rule applied in the derivation of the left premiss. By examining all possible rule applications, we show that the application of $\mathsf{cut}$ can be replaced by one or more applications of $\mathsf{cut}$ at a smaller height. For instance, assume that the last rule applied is $\mathsf{L}\Box$.

$$\dfrac{\mathsf{L}\Box\,\dfrac{G \mid \langle A\rangle, \Box A, \Gamma \Rightarrow \Delta, \bot}{G \mid \Box A, \Gamma \Rightarrow \Delta, \bot} \qquad G \mid \bot, \Box A, \Gamma \Rightarrow \Delta}{G \mid \Box A, \Gamma \Rightarrow \Delta}\,\mathsf{cut}$$

The derivation is transformed as follows, with a height-preserving application of $\mathsf{Lwk}$ and an application of $\mathsf{cut}$ of smaller height.

$$\dfrac{\dfrac{G \mid \langle A\rangle, \Box A, \Gamma \Rightarrow \Delta, \bot \qquad \dfrac{G \mid \bot, \Box A, \Gamma \Rightarrow \Delta}{G \mid \bot, \langle A\rangle, \Box A, \Gamma \Rightarrow \Delta}\,\mathsf{Lwk}}{G \mid \langle A\rangle, \Box A, \Gamma \Rightarrow \Delta}\,\mathsf{cut}}{G \mid \Box A, \Gamma \Rightarrow \Delta}\,\mathsf{L}\Box$$

The situation is similar if the last rule in the derivation of the left premiss is applied to some sequent in $G$.

(ii) The cut formula $\bot$, $\top$, or $p$ is not principal in the last rule applied in the derivation of the right premiss. The case is analogous to (i). As an example, suppose that the last rule applied is $\mathsf{R}\Box\mathsf{m}$.

$$\dfrac{G \mid \langle \Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B, \bot \qquad \dfrac{G \mid \bot, \langle \Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B \mid \Sigma \Rightarrow B}{G \mid \bot, \langle \Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B}\,\mathsf{R}\Box\mathsf{m}}{G \mid \langle \Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B}\,\mathsf{cut}$$

The derivation is converted into

$$\dfrac{\dfrac{\mathsf{Ewk}\,\dfrac{G \mid \langle \Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B, \bot}{G \mid \langle \Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B, \bot \mid \Sigma \Rightarrow B} \qquad G \mid \bot, \langle \Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B \mid \Sigma \Rightarrow B}{G \mid \langle \Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B \mid \Sigma \Rightarrow B}\,\mathsf{cut}}{G \mid \langle \Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B}\,\mathsf{R}\Box\mathsf{m}$$

where $\mathsf{cut}$ is applied at a smaller height.

(iii) The cut formula $\bot$, $\top$, or $p$ is principal in the last rule applied in the derivation of both premisses. Then the cut formula is $p$, as $\bot$ (resp. $\top$) is never principal on the right-hand side (resp. the left-hand side) of the conclusion of any rule application. This means that both premisses are derived by $\mathsf{init}$, which implies that $h = 0$. Then we are back to case **(A)**.

**(C)** Assume $\forall h Cut(c, h)$. We prove that all applications of $\mathsf{sub}$ where $A$ has weight $c$ are admissible. The proof is by induction on the height $m$ of the derivation of $G \mid \langle A^{n_1}, \Pi_1\rangle, \ldots, \langle A^{n_k}, \Pi_k\rangle, \Gamma \Rightarrow \Delta$. If $m = 0$ or no block among $\langle A, \Pi_1\rangle, \ldots, \langle A, \Pi_k\rangle$ is principal in the last rule application, then the

proof proceeds similarly to previous cases. If $m > 0$ and at least one block among $\langle A, \Pi_1\rangle, \ldots, \langle A, \Pi_k\rangle$ is principal in the last rule application we have the following possibilities.

• The last rule applied is $\mathsf{R}\Box$:

$$\dfrac{\overset{①}{G \mid \langle A^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D \mid A^{n_i}, \Pi_i \Rightarrow D} \qquad \begin{array}{c}\{G \mid \langle A^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D \mid D \Rightarrow C\}_{C\in\Pi_i} \\ \{G \mid \langle A^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D \mid D \Rightarrow A\}_1^{n_i} \quad \vdots\end{array}}{G \mid \langle A^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D}\ \mathsf{R}\Box$$

The derivation is converted as follows. First we derive:

$$\dfrac{\dfrac{G \mid \Sigma \Rightarrow A}{G \mid \Sigma \Rightarrow A \mid A^{n_i}, \Pi_i \Rightarrow D}\ \mathsf{Ewk} \qquad \left\{\dfrac{G \mid A \Rightarrow B}{G \mid A \Rightarrow B \mid A^{n_i}, \Pi_i \Rightarrow D}\ \mathsf{Ewk}\right\}_{B\in\Sigma} \qquad ①}{G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D \mid A^{n_i}, \Pi_i \Rightarrow D}\ \mathsf{sub}$$

(where rule $\mathsf{sub}$ possibly applies to further blocks inside $\Gamma$). Then by applying $\mathsf{Ewk}$ to $G \mid \Sigma \Rightarrow A$ we obtain $G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D \mid \Sigma \Rightarrow A$. By auxiliary applications of $\mathsf{wk}$ we can cut $A$ and get $G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D \mid \Sigma, A^{n_i-1}, \Pi_i \Rightarrow D$. Then with further applications of cut (each time with auxiliary applications of $\mathsf{wk}$) we obtain $G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D \mid \Sigma^{n_i}, \Pi_i \Rightarrow D$. By doing the same with the other premisses of $\mathsf{R}\Box$ in the initial derivation we obtain also $\{\{G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D \mid D \Rightarrow B\}_{B\in\Sigma}\}_1^{n_1}$ and $\{G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D \mid D \Rightarrow C\}_{C\in\Pi_i}$. Finally by $\mathsf{R}\Box$ we derive the conclusion of $\mathsf{sub}$ $G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D$.

• The last rule applied is $\mathsf{C}$:

$$\dfrac{G \mid \langle A^{n_i}, \Pi_i\rangle, \langle A^{n_j}, \Pi_j\rangle, \langle A^{n_i}, A^{n_j}, \Pi_i, \Pi_j\rangle, \Gamma \Rightarrow \Delta}{G \mid \langle A^{n_i}, \Pi_i\rangle, \langle A^{n_j}, \Pi_j\rangle, \Gamma \Rightarrow \Delta}\ \mathsf{C}$$

By applying $\mathsf{sub}$ to the premiss we obtain
$G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \langle \Sigma^{n_j}, \Pi_j\rangle, \langle \Sigma^{n_i}, \Sigma^{n_j}, \Pi_i, \Pi_j, \rangle, \Gamma \Rightarrow \Delta$, then by $\mathsf{C}$ we derive
$G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \langle \Sigma^{n_j}, \Pi_j\rangle, \Gamma \Rightarrow \Delta$.

• The last rule applied is $\mathsf{T}$:

$$\dfrac{G \mid A^{n_i}, \Pi_i, \langle A^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta}{G \mid \langle A^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta}\ \mathsf{T}$$

By applying the inductive hypothesis to the premiss we obtain
$G \mid A^{n_i}, \Pi_i, \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta$. Then, from this and $G \mid \Sigma \Rightarrow A$, by several applications of $\mathsf{cut}$ (each time with auxiliary applications of $\mathsf{wk}$) we obtain $G \mid \Sigma^{n_i}, \Pi_i, \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta$. Finally, by $\mathsf{T}$ we derive $G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta$.

• The last rule applied is $\mathsf{P}$:

$$\dfrac{G \mid \langle A^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta \mid A^{n_i}, \Pi_i \Rightarrow}{G \mid \langle A^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta}\ \mathsf{P}$$

By applying the inductive hypothesis to the premiss (after auxiliary applications of $\mathsf{Ewk}$ to the other premisses of $\mathsf{sub}$) we obtain $G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta \mid A^{n_i}, \Pi_i \Rightarrow$. Then, from this and $G \mid \Sigma \Rightarrow A$, by several applications of $\mathsf{cut}$ (each time with auxiliary applications of $\mathsf{wk}$) we obtain $G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta \mid \Sigma^{n_i}, \Pi_i \Rightarrow$. Finally, by $\mathsf{P}$ we derive $G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta$.

• The last rule applied is $\mathsf{D}_2$: Then $G \mid \langle A^{n_i}, \Pi_i\rangle, \langle A^{n_j}, \Pi_j\rangle, \Gamma \Rightarrow \Delta$ has been derived by the following premisses. $G \mid \langle A^{n_i}, \Pi_i\rangle, \langle A^{n_j}, \Pi_j\rangle, \Gamma \Rightarrow \Delta \mid A^{n_i}, A^{n_j}, \Pi_i, \Pi_j \Rightarrow$; $\{G \mid \langle A^{n_i}, \Pi_i\rangle, \langle A^{n_j}, \Pi_j\rangle, \Gamma \Rightarrow \Delta \mid\Rightarrow A, A\}_1^{n_i+n_j}$; $\{\{G \mid \langle A^{n_i}, \Pi_i\rangle, \langle A^{n_j}, \Pi_j\rangle, \Gamma \Rightarrow \Delta \mid\Rightarrow A, C\}_{C\in\Pi_i}\}_1^{n_j}$; $\{\{G \mid \langle A^{n_i}, \Pi_i\rangle, \langle A^{n_j}, \Pi_j\rangle, \Gamma \Rightarrow \Delta \mid\Rightarrow A, D\}_{D\in\Pi_j}\}_1^{n_i}$; and $\{G \mid \langle A^{n_i}, \Pi_i\rangle, \langle A^{n_j}, \Pi_j\rangle, \Gamma \Rightarrow \Delta \mid\Rightarrow A, A\}_{C\in\Pi_i, D\in\Pi_j}$. We consider the other premisses of $\mathsf{sub}$ and apply $\mathsf{cut}$ many times (each time with auxiliary applications of $\mathsf{wk}$) so to replace all occurrences of $A$ with formulas in $\Sigma$. As final step we can apply $\mathsf{D}_2$ and obtain $G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \langle \Sigma^{n_j}, \Pi_j\rangle, \Gamma \Rightarrow \Delta$.

• The remaining cases $\mathsf{D}_n^+$ and $\mathsf{D}_1$ are similar to the previous ones.

**(D)** Assume $\forall c' < c.\,(Sub(c') \wedge \forall h'.\,Cut(c', h'))$ and $\forall h'' < h.\,Cut(c, h'')$. We show that all applications of $\mathsf{cut}$ of height $h$ on a cut formula of weight $c$ can be replaced by different applications of $\mathsf{cut}$, either of smaller height or on a cut formula of smaller weight. We can assume $h, c > 0$ as the cases $h = 0$ and $c = 0$ have been already considered in **(A)** and **(B)**. We distinguish two cases.

(i) The cut formula is not principal in the last rule application in the derivation of at least one of the two premisses of $\mathsf{cut}$. This case is analogous to (i) or (ii) in **(B)**.

(ii) The cut formula is principal in the last rule application in the derivation of both premisses. Then the cut formula is either $B \circ C$, with $\circ \in \{\wedge, \vee, \rightarrow\}$, or $\Box B$.

• The case of boolean connective is standard. We consider as an example $B \rightarrow C$. We have:

$$\dfrac{\mathsf{R}\rightarrow\dfrac{\overset{①}{G \mid B, \Gamma \Rightarrow \Delta, B \rightarrow C, C}}{G \mid \Gamma \Rightarrow \Delta, B \rightarrow C} \qquad \dfrac{\overset{②}{G \mid B \rightarrow C, \Gamma \Rightarrow \Delta, B} \qquad \overset{③}{G \mid C, B \rightarrow C, \Gamma \Rightarrow \Delta}}{G \mid B \rightarrow C, \Gamma \Rightarrow \Delta}\,\mathsf{L}\rightarrow}{G \mid \Gamma \Rightarrow \Delta}\,\mathsf{cut}$$

The derivation is converted into the following one:

$$\dfrac{\mathsf{cut}\dfrac{\mathsf{Rwk}\dfrac{\mathsf{Rwk}\dfrac{G \mid \Gamma \Rightarrow \Delta, B \rightarrow C}{G \mid \Gamma \Rightarrow \Delta, B \rightarrow C, B}}{G \mid \Gamma \Rightarrow \Delta, B}}{G \mid \Gamma \Rightarrow \Delta, B, C}\,② \qquad \dfrac{①\quad \dfrac{\mathsf{Rwk}\dfrac{\mathsf{Lwk}\dfrac{G \mid B \rightarrow C, \Gamma \Rightarrow \Delta}{G \mid B, B \rightarrow C, \Gamma \Rightarrow \Delta}}{G \mid B, B \rightarrow C, \Gamma \Rightarrow \Delta, C}}{G \mid B, \Gamma \Rightarrow \Delta, C}\,\mathsf{cut}}{G \mid \Gamma \Rightarrow \Delta, C}\,\mathsf{cut} \qquad \mathsf{cut}\dfrac{\mathsf{Lwk}\dfrac{G \mid \Gamma \Rightarrow \Delta, B \rightarrow C}{G \mid C, \Gamma \Rightarrow \Delta, B \rightarrow C}}{G \mid C, \Gamma \Rightarrow \Delta}\,③}{G \mid \Gamma \Rightarrow \Delta}\,\mathsf{cut}$$

• If the cut formula is $\Box B$ we have

$$G \mid \langle \Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B \mid \Sigma \Rightarrow B$$

$$\dfrac{\mathsf{R}\Box\dfrac{\vdots \quad \{G \mid \langle \Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B \mid B \Rightarrow C\}_{C\in\Sigma}}{G \mid \langle \Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B} \qquad \dfrac{G \mid \langle B\rangle, \Box B, \langle \Sigma\rangle, \Gamma \Rightarrow \Delta}{G \mid \Box B, \langle \Sigma\rangle, \Gamma \Rightarrow \Delta}\,\mathsf{L}\Box}{G \mid \langle \Sigma\rangle, \Gamma \Rightarrow \Delta}\,\mathsf{cut}$$

The derivation is converted as follows, with several applications of $\mathsf{cut}$ of smaller height and an admissible application of $\mathsf{sub}$.

$$\dfrac{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B \mid \Sigma \Rightarrow B \qquad \dfrac{G \mid \Box B, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta}{G \mid \Box B, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow B}\,\mathsf{Ewk}}{④\ G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow B}\,\mathsf{cut}$$

$$\dfrac{\dfrac{\dfrac{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B}{G \mid \langle B\rangle, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B}\,\mathsf{Lwk} \qquad G \mid \langle B\rangle, \Box B, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta}{G \mid \langle B\rangle, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta}\,\mathsf{cut}}{⑤\ G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta \mid \langle B\rangle, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta}\,\mathsf{Ewk}$$

$$\dfrac{\dfrac{\dfrac{④ \quad \left\{\dfrac{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B \mid B \Rightarrow C \qquad \dfrac{G \mid \Box B, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta}{G \mid \Box B, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta \mid B \Rightarrow C}\,\mathsf{Ewk}}{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta \mid B \Rightarrow C}\,\mathsf{cut}\right\}_{C \in \Sigma} \quad ⑤}{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta \mid \langle\Sigma\rangle, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta}\,\mathsf{sub}}{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta}\,\mathsf{Lctr}}{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta}\,\mathsf{Ectr}$$

□

Given the admissibility of the structural rules and cut we can prove that the calculi are syntactically complete with respect to the corresponding axiom systems.

**Theorem 4.5** (Syntactic completeness)**.** If $\vdash_{\mathbf{E}^*} A$, then $\Rightarrow A$ is derivable in $\mathbf{H.E}^*$.

*Proof.* As usual, we have to show that all axioms of $\mathbf{E}^*$ are derivable in $\mathbf{H.E}^*$, and that all rules of $\mathbf{E}^*$ are admissible in $\mathbf{H.E}^*$. The derivations of the modal axioms and rules are displayed in Figure 5. For the derivations of the axioms we implicitly consider Proposition 4.1. For the derivation of rule $RE$ we assume that $A \to B$ and $B \to A$ are derivable in $\mathbf{E}^*$, and for the derivation of rule $RD_n^+$ we assume that $\neg(A_1, \ldots, A_n)$ is derivable in $\mathbf{ED_n^+}^*$. Finally, $MP$ is simulated by cut in the usual way. □

# 5 Complexity of proof search

One of the advantages of formal calculi is that they can often be used to establish complexity-optimal decision procedures for the corresponding logics via backwards proof search. In this section we will use our hypersequent calculi to do so. Before considering the results in detail, we note again that since all the considered logics have standard sequent calculi, generic PSPACE complexity results for all the logics follow standard backwards proof search using these calculi. However, as established in [40], in many cases dropping the axiom $C$ lowers the complexity of the logic to coNP. Here we show how the hypersequent calculi give rise to complexity optimal decision procedures for the logics without $C$, before briefly commenting on the case where $C$ is present.

## Extensions without axiom $C$

The decision procedures for the logics without the axiom $C$ implement backwards proof search on a polynomially bounded nondeterministic Turing machine

with universal choices to handle the branching caused by rules with several premisses. Since all the rules are invertible, we can fix an order in which the rules are applied. To prevent loops, we employ a *local loop checking* strategy, stating that a rule is not applied (bottom-up) to a hypersequent $G$, if at least one of its premisses is trivial in the sense that each of its components can be derived from a component of the conclusion using only weakening and contraction. The formal definition is as follows.

**Definition 5.1.** An application of a hypersequent rule with premisses $H_1, \ldots, H_n$ and conclusion $G$ satisfies the *local loop checking condition* if for each premiss $H_i$ there exists a component $\Gamma \Rightarrow \Delta$ in $H_i$ such that for no component $\Sigma \Rightarrow \Pi$ of the conclusion $G$ we have: for all $A \in \Gamma$ also $A \in \Sigma$; and for all $\langle\Theta\rangle \in \Gamma$ there is a $\langle\Xi\rangle \in \Sigma$ with $\mathsf{set}(\Theta) = \mathsf{set}(\Xi)$; and $\mathsf{set}(\Delta) \subseteq \mathsf{set}(\Pi)$.

Note that in order to check whether a rule satisfies the local loop checking condition, it suffices to perform a check on every pair consisting of the conclusion and one of the premisses, and that each of these checks can be performed in time polynomial in the size of the premiss and the conclusion. Hence in the proof search we can absorb the local loop check into the universal choices over the premisses of branching rules.

Since the rules are cumulative, every application of a rule satisfying the local loop checking condition adds in each of its premisses at least one new block or formula to an existing component, or adds a new component, which is not subsumed by a component of the conclusion. The following proposition shows that local loop checking does not jeopardise completeness.

**Proposition 5.1.** If a hypersequent is derivable in $\mathbf{H.E}^*$ with a derivation of height $n$, then it is derivable using a derivation of height $n$ in which every rule application satisfies the local loop checking condition.

*Proof.* By induction on the height of the derivation. The zero-premiss rules trivially satisfy the local loop checking condition. If the height of the derivation is $n+1$, consider the bottom-most rule application. If it satisfies the local loop checking condition, we apply the induction hypothesis to each of its premisses and are done. Otherwise, there is a premiss such that for each of its components $\Gamma, \langle\Theta_1\rangle, \ldots, \langle\Theta_m\rangle \Rightarrow \Delta$ (where $\Gamma$ does not contain any block) there is a component $\Sigma \Rightarrow \Pi$ of the conclusion $G$ of the derivation with $\mathsf{set}(\Gamma) \subseteq \mathsf{set}(\Sigma)$, and for every $i \leq m$ there is a $\langle\Theta'_i\rangle \in \Sigma$ with $\mathsf{set}(\Theta_i) = \mathsf{set}(\Theta'_i)$, and $\mathsf{set}(\Delta) \subseteq \mathsf{set}(\Pi)$. Using height-preserving admissibility of the structural rules (Proposition 4.3) we thus obtain a derivation of $G$ of height $n$, and an appeal to the induction hypothesis yields a derivation of height $n$ where every rule application satisfies the local loop checking condition. □

Note that in the proof of this proposition, no new rule applications are added to a derivation, and that the order of rule applications is preserved in the proof of admissibility of the structural rules (Proposition 4.3). Hence given a derivation of a hypersequent, we can first adjust the ordering of the rules using invertibility, then remove all rule applications violating the local loop checking condition. This yields completeness of proof search under these constraints:

**Corollary 5.2.** Proof search in $\mathbf{H.E}^*$ with local loop checking and a fixed order on the applications of rules is complete. □

**Algorithm 1:** Decision procedure for the derivability problem in $\mathbf{H}.\mathbf{E}^*$

**Input:** A hypersequent $G$ and the code of a logic $\mathbf{L}$
**Output:** "yes" if $G$ is derivable in $\mathbf{H}.\mathbf{L}$, a hypersequent if it is not.

```
if there is a component Γ ⇒ Δ in G with ⊥ ∈ Γ, or ⊤ ∈ Δ, or
  Γ ∩ Δ ≠ ∅ then
    return "yes" and halt ;
else if there is an applicable rule then
    pick the first applicable rule;
    universally choose a premiss H of this rule application;
    check that the premiss does not violate the local loop checking
      conditions;
    check recursively whether H is derivable, output the answer and
      halt;
else
    return G and halt;
end
```

The proof search algorithm thus applies the rules backwards in an arbitrary but fixed order, universally chooses one of their premisses and then recursively checks that this premiss is derivable. The procedure is shown in Algorithm 1. In order to facilitate the countermodel construction for underivable hypersequents in the next section, we show termination for all considered logics, even those containing axiom $C$:

**Theorem 5.3.** Algorithm 1 terminates for all logics $\mathbf{H}.\mathbf{E}^*$.

*Proof.* Due to the subformula property of the rules, every formula occurring in a hypersequent in a run of Algorithm 1 is a subformula of the input. Moreover, local loop checking prevents the duplication of formulas, blocks and components. Thus, every component occurring in a run of the algorithm contains a subset of (occurrences of) subformulas of the input both on its antecedent and consequent, together with a set of blocks, each containing a subset of (occurrences of) subformulas of the input. Since there are only finitely many of these, the number of possible components is finite, and hence also the number of hypersequents occurring in a run of the algorithm. Since every rule application satisfying local loop checking strictly increases the size of the hypersequent, each run of the algorithm thus halts after finitely many steps. □

For the logics without axiom $C$, a closer analysis of the run time yields the optimal complexity bound:

**Theorem 5.4.** For the logics without $C$, Algorithm 1 runs in coNP, whence for these logics the calculi provide a complexity-optimal decision procedure.

*Proof.* Since the procedure is in the form of a non-deterministic Turing machine with universal choices, it suffices to show that every computation of this machine has polynomial length. Every application of a rule adds either a subformula of its conclusion or a new block to one of the components, or adds a new component. Due to local loop checking it never adds a formula, block or component which

is already in the conclusion, so it suffices to calculate the maximal size of a hypersequent occurring in proof search for $G$. Suppose that the size of $G$ is $n$. Then both the number of components in $G$ and the number of subformulas of $G$ are bounded by $n$. Since the local loop check prevents the duplication of formulas, each component contains at most $n$ formulas in the antecedent and $n$ formulas in the consequent. Moreover, since we only consider logics without the axiom $C$, every newly created block contains exactly one formula. Again, due to the local loop checking condition no block is duplicated, so every component contains at most $n$ blocks. Thus every component has size at most $3n$. The procedure creates new components from a block and a formula of an already existing component using one of the rules $\mathsf{R}\Box$ and $\mathsf{R}\Box\mathsf{m}$, or from $\ell$ components using one of the rules $\mathsf{P}$, $\mathsf{D}_2$, $\mathsf{D}_1$, $\mathsf{D}^+_\ell$, with $\ell \leq k$ for a fixed $k$ depending on the logic. Hence there are at most $n^2 + k \cdot n^k$ many different components which can be created without violating the local loop checking condition. Thus every hypersequent occurring in the proof contains at most $n + n^2 + k \cdot n^k$ many components, each of size at most $3n$, giving a total size of $\mathcal{O}(n^3)$, resp. $\mathcal{O}(n^{k+1})$ for $k > 2$. Since moreover checking whether a pair of premiss and conclusion of a rule violates the local loop checking conditions can be done in time polynomial in the combined size of premiss and conclusion, the total running time is polynomial. $\Box$

As noted above, Algorithm 1 works properly also for logics with the axiom $C$, ensuring in particular termination. However, hypersequents occurring in a proof of $H$ can be exponentially large with respect to the size of $H$. This is due to the presence of the rule $\mathsf{C}$ that, given $n$ formulas $\Box A_1, \ldots, \Box A_n$, allows one to build a block for every subset of $\{A_1, \ldots, A_n\}$. In this respect, this decision procedure does not match the $\mathsf{PSPACE}$ complexity upper bound established for these systems by Vardi [40]. However, this is not really unexpected, since one of the main appeals of the hypersequent calculi is that they can be used to directly construct countermodels for unprovable hypersequents, and in some logics with $C$ it is possible to force exponentially large countermodels, in particular in normal modal logic $\mathbf{K}$ [3]. Hence for these logics the hypersequents will need to be of exponential size, suggesting that we need to modify the hypersequent calculi to obtain complexity-optimal decision procedures, as explained next.

## Logics with axiom $C$

In order to obtain a $\mathsf{PSPACE}$ decision procedure for logics with axiom $C$ we must adopt a different strategy. Since already the standard sequent calculi could be used to obtain complexity-optimal decision procedures in a standard way, we only sketch the ideas. Instead of the rules in Figure 4, we consider their *un-Kleene'd* – and non-invertible – version, i.e., the ones with all principal formulas and structures deleted from the premisses. For instance $\mathsf{R}\Box\mathsf{m}$, $\mathsf{R}\Box$ and $\mathsf{C}$ are replaced respectively with

$$\frac{G \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow B}{G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Box B, \Delta} \qquad \frac{G \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow B \quad \{G \mid \Gamma \Rightarrow \Delta \mid B \Rightarrow A\}_{A \in \Sigma}}{G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Box B, \Delta} \qquad \frac{G \mid \Gamma, \langle\Sigma, \Pi\rangle \Rightarrow \Delta}{G \mid \Gamma, \langle\Sigma\rangle, \langle\Pi\rangle \Rightarrow \Delta}$$

Call the resulting calculus $\mathbf{H.E}^*_{_}$. Backwards proof search is then implemented on an alternating Turing machine by existentially guessing the last applied rule except for $\mathsf{N}$, and universally checking that all of its premisses are derivable. To

ensure that $\mathsf{N}$ is applied if it is present in the system, we stipulate that it is applied once to every component of the input, and that if the existentially guessed rule creates a new component, the rule $\mathsf{N}$ is applied immediately afterwards to each of its premisses. Since no rule application keeps the principal formulas in the premisses, and since the rule $\mathsf{N}$ if present is applied exactly once to every component, there is no need for any loop checking condition.

The calculi $\mathbf{H}.\mathbf{E}^*_{_}$ are sound and complete. Soundness is obvious, since we can add the missing formulas and structures and recover derivations in $\mathbf{H}.\mathbf{E}^*$. Completeness is seen easiest by simulating the standard sequent calculi, e.g. [25]. We can show that the calculi $\mathbf{H}.\mathbf{E}^*_{_}$ give a $\mathsf{PSPACE}$ upper bound.

**Theorem 5.5.** Backwards proof search in $\mathbf{H}.\mathbf{E}^*_{_}$ is in $\mathsf{PSPACE}$.

*Proof.* We need to show that every run of the procedure terminates in polynomial time. Assume that the size of the input is $n$. Let the *weight* of a component in a hypersequent be the sum of the weights of the formulas and blocks occurring in it according to Definition 4.3, and suppose that the maximal weight of components in the input is $w$. Then every rule apart from $\mathsf{N}$ decreases the weight of the component active in its conclusion. Moreover, a new component is only introduced in place of at least one subformula of the input, hence any hypersequent occurring in the proof search has at most $n + n$ components. The weight of each of these components is at most the maximal weight of a component of the input (plus one in the cases with $\mathsf{N}$). Since the rule $\mathsf{N}$ is applied at most once to each component, it is thus applied at most $n$ times in the total proof search. Thus the runtime in total is $\mathcal{O}(n^2 \cdot w)$, hence polynomial in the size of the input. Thus the procedure runs in alternating polynomial time, and thus in $\mathsf{PSPACE}$. □

Thus, the situation of logics with axiom $C$ can be summarised as follows. On the one hand, we have a fully invertible calculus $\mathbf{H}.\mathbf{EC}^*$ which is terminating but not optimal. As we shall see in the next section, this calculus allows for direct extraction of countermodels from single failed proofs. On the other hand, we have a calculus $\mathbf{H}.\mathbf{EC}^*{}_{_}$ which is optimal but not invertible, whence direct extraction of countermodels from single failed proofs is not possible. As for many other logics, this illustrates the existence of a necessary trade-off between the optimal complexity of the calculus and the direct countermodel extraction.

# 6 Countermodel extraction and semantic completeness

We now prove semantic completeness of the hypersequent calculi, i.e., every valid hypersequent is derivable. This amounts to showing that a non-provable hypersequent has a countermodel. Countermodels are found in the bi-neighbourhood semantics, as it is better suited for direct countermodel extraction from failed proofs than the standard semantics. The reason is that, in order to define a standard neighbourhood model, we need to determine *exactly* the truth sets of formulas: If we want a world $w$ to force $\Box A$, then we have to make sure that $[\![A]\!]$ belongs to $\mathcal{N}(w)$, thus $[\![A]\!]$ must be computed. But this is in conflict with the fact that failed proofs only provide *partial information*. Intuitively, countermodel extraction from a saturated hypersequent in a proof of $H$ is based on

the natural semantic reading according to which every component corresponds to a world in the model, and every formula in the antecedent (respectively in the consequent) of a component is true (respectively false) in the corresponding world. But, unless one resorts to a form of the analytic cut rule as implicitly done in [25], it is hardly ever the case that every subformula of $H$ is either in the antecedent or in the consequent of every component, thus the failed proof does not suffice to exactly determine the truth set of every subformula. On the contrary, in order for a world $w$ to force $\Box A$ in a bi-neighbourhood model it suffices to find a suited pair $(\alpha, \beta)$ such that $\alpha \subseteq [\![A]\!] \subseteq \mathcal{W} \setminus \beta$. As we shall see, such a pair can be extracted direclty from the failed proof even without knowing exactly the extension of $[\![A]\!]$. This is done in the simplest way: limiting the explanation to unitary blocks, each block $\langle A \rangle$ "occurring" in a world (i.e., in the antecedent of a component of a hypesequent) will determine a bi-neighbourhood pair $(\alpha, \beta)$ of that world, where $\alpha$ collects worlds which force $A$ (i.e., having $A$ in their antecedent) and $\beta$ collects worlds refuting $A$ (i.e., having $A$ in their consequent). Since a world may not contain $A$ neither in its antecedent, nor in its consequent, $\alpha$ and $\beta$ are not expected to be complementary. That is why bi-neighbourhood models can be seen as a kind of "partial" neighbourhood models (in the standard sense) arising in a natural way from a saturared hypersequent. The situation is analogous to *partial* models associated to sets of signed formulas determined by saturated branches in Smullyan's tableaux.

In order to prove semantic completeness we make use of the backwards proof search strategy based on local loop checking already considered in Section 5 (Algorithm 1). This strategy amounts to considering the following notion of saturation, stating that no bottom-up rule application is allowed to initial sequents, and that a bottom-up application of a rule $R$ is not allowed to a hypersequent $G$ if $G$ already fulfills the corresponding saturation condition $(R)$.

**Definition 6.1** (Saturated hypersequent)**.** Let $H = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ be a hypersequent occurring in a proof for $H'$. The saturation conditions associated to each application of a rule of $\mathbf{H.E}^*$ are as follows: $(\mathsf{init})$ $\Gamma_i \cap \Delta_i = \emptyset$. $(\mathsf{L}\bot)$ $\bot \notin \Gamma_i$. $(\mathsf{R}\top)$ $\top \notin \Delta_i$. $(\mathsf{L}\to)$ If $A \to B \in \Gamma_i$, then $A \in \Delta_i$ or $B \in \Gamma_i$. $(\mathsf{R}\to)$ If $A \to B \in \Delta_i$, then $A \in \Gamma_i$ and $B \in \Delta_i$. $(\mathsf{L}\wedge)$ If $A \wedge B \in \Gamma_i$, then $A \in \Gamma_i$ and $B \in \Gamma_i$. $(\mathsf{R}\wedge)$ If $A \wedge B \in \Delta_i$, then $A \in \Delta_i$ or $B \in \Delta_i$. $(\mathsf{L}\Box)$ If $\Box A \in \Gamma_i$, then $\langle A \rangle \in \Gamma_i$. $(\mathsf{R}\Box)$ If $\langle \Sigma \rangle, \Gamma \Rightarrow \Delta, \Box B$ is in $H$, then there is $\Gamma' \Rightarrow \Delta', B$ in $H$ such that $\mathsf{set}(\Sigma) \subseteq \Gamma'$, or there is $B, \Gamma' \Rightarrow \Delta', A$ in $H$ for some $A \in \Sigma$. $(\mathsf{R}\Box\mathsf{m})$ If $\langle \Sigma \rangle, \Gamma \Rightarrow \Delta, \Box B$ is in $H$, then there is $\Gamma' \Rightarrow \Delta', B$ in $H$ such that $\mathsf{set}(\Sigma) \subseteq \Gamma'$. $(\mathsf{N})$ $\langle \top \rangle \in \Gamma_i$. $(\mathsf{C})$ If $\langle \Sigma \rangle, \langle \Pi \rangle \in \Gamma_i$, then there is $\langle \Omega \rangle \in \Gamma_i$ such that $\mathsf{set}(\Sigma, \Pi) = \mathsf{set}(\Omega)$. $(\mathsf{T})$ If $\langle \Sigma \rangle \in \Gamma_n$, then $\mathsf{set}(\Sigma) \subseteq \Gamma_n$. $(\mathsf{P})$ If $\Gamma, \langle \Sigma \rangle \Rightarrow \Delta$ is in $H$, then there is $\Gamma' \Rightarrow \Delta'$ in $H$ such that $\mathsf{set}(\Sigma) \subseteq \Gamma'$. $(\mathsf{D}_1)$ If $\Gamma, \langle \Sigma \rangle \Rightarrow \Delta$ is in $H$, then there is $\Gamma' \Rightarrow \Delta'$ in $H$ such that $\mathsf{set}(\Sigma) \subseteq \Gamma'$, or there is $\Gamma' \Rightarrow \Delta', A$ in $H$ for some $A \in \Sigma$. $(\mathsf{D}_2)$ If $\Gamma, \langle \Sigma \rangle, \langle \Pi \rangle \Rightarrow \Delta$ is in $H$, then there is $\Gamma' \Rightarrow \Delta'$ in $H$ such that $\mathsf{set}(\Sigma, \Pi) \subseteq \Gamma'$, or there is $\Gamma' \Rightarrow \Delta', A, B$ in $H$ for some $A \in \Sigma, B \in \Pi$. $(\mathsf{D}_n^+)$ If $\Gamma, \langle \Sigma_1 \rangle, \ldots, \langle \Sigma_n \rangle \Rightarrow \Delta$ is in $H$, then there is $\Gamma' \Rightarrow \Delta'$ in $H$ such that $\mathsf{set}(\Sigma_1, \ldots, \Sigma_n) \subseteq \Gamma'$.

We say that $H$ is *saturated* with respect to an application of a rule $R$ if it satisfies the saturation condition $(R)$ for that particular rule application, and that it is saturated with respect to $\mathbf{H.E}^*$ if it is saturated with respect to all possible applications of any rule of $\mathbf{H.E}^*$.

**Proposition 6.1.** If Algorithm 1 with input $G$ returns a hypersequent $H$, then

$H$ is saturated and, for every component $\Gamma \Rightarrow \Delta$ of $G$, there is a component $\Sigma \Rightarrow \Pi$ of $H$ with $\mathsf{set}(\Gamma) \subseteq \mathsf{set}(\Sigma)$ and $\mathsf{set}(\Delta) \subseteq \mathsf{set}(\Pi)$.

*Proof.* Saturation of $H$ follows from verifying that if one of the saturation conditions is not met, the corresponding rule can be applied without violating the local loop checking condition. Since Algorithm 1 applies all possible rules satisfying the local loop checking condition before halting and returning a hypersequent, $H$ must be saturated. The second statement follows from the cumulative nature of the rules. □

Then, given a saturated hypersequent $H$ we can directly construct a countermodel for $H$ in the bi-neighbourhood semantics in the following way.

**Definition 6.2** (Countermodel construction)**.** Let $H$ be a saturated hypersequent occurring in a proof for $H'$. Moreover, let $e : \mathbb{N} \longrightarrow H$ be an enumeration of the components of $H$. Given $e$, we can write $H$ as $\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_k \Rightarrow \Delta_k$. We call $k$ the *length* of $H$. The model $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$ is defined as follows:

- $\mathcal{W} = \{n \mid \Gamma_n \Rightarrow \Delta_n \in H\}$.
- $\mathcal{V}(p) = \{n \mid p \in \Gamma_n\}$.
- For all blocks $\langle \Sigma \rangle$ appearing in a component $\Gamma_m \Rightarrow \Delta_m$ of $H$, $\Sigma^+ = \{n \mid \mathsf{set}(\Sigma) \subseteq \Gamma_n\}$ and $\Sigma^- = \{n \mid \Sigma \cap \Delta_n \neq \emptyset\}$.
- The definition of $\mathcal{N}$ depends whether the calculus is or not monotonic:
  - Non-monotonic case: $\mathcal{N}(n) = \{(\Sigma^+, \Sigma^-) \mid \langle \Sigma \rangle \in \Gamma_n\}$.
  - Monotonic case: $\mathcal{N}(n) = \{(\Sigma^+, \emptyset) \mid \langle \Sigma \rangle \in \Gamma_n\}$.

**Lemma 6.2.** Let $H = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_k \Rightarrow \Delta_k$ be a saturated hypersequent, and $\mathcal{M}$ be the model defined on the basis of $H$ as in Definition 6.2. Then for every $A$, $\langle \Sigma \rangle$ and every $n \in \mathcal{W}$, we have:

- if $A \in \Gamma_n$, then $\mathcal{M}, n \Vdash A$;
- if $\langle \Sigma \rangle \in \Gamma_n$, then $\mathcal{M}, n \Vdash \Box \bigwedge \Sigma$; and
- if $A \in \Delta_n$, then $\mathcal{M}, n \nVdash A$.

Moreover, if the proof is in calculus $\mathbf{H.EX}^*$, then $\mathcal{M}$ is a X-model.

*Proof.* The first claim is proved by mutual induction on $A$ and $\langle \Sigma \rangle$.

($p \in \Gamma_n$) By definition, $n \in \mathcal{V}(p)$. Thus $n \Vdash p$.

($p \in \Delta_n$) By saturation of $\mathsf{init}$, $p \notin \Gamma_n$. Then $n \notin \mathcal{V}(p)$, thus $n \nVdash p$.

($B \wedge C \in \Gamma_n$) By saturation of $\mathsf{L}\wedge$, $B \in \Gamma_n$ and $C \in \Gamma_n$. Then by inductive hypothesis, $n \Vdash B$ and $n \Vdash C$, thus $n \Vdash B \wedge C$.

($B \wedge C \in \Delta_n$) By saturation of $\mathsf{R}\wedge$, $B \in \Delta_n$ or $C \in \Delta_n$. Then by inductive hypothesis, $n \nVdash B$ or $n \nVdash C$, thus $n \nVdash B \wedge C$.

For $A = B \vee C, B \to C$, the proof is similar to the previous cases.

($\langle \Sigma \rangle \in \Gamma_n$) In the non-monotonic case we have: By definition $(\Sigma^+, \Sigma^-) \in \mathcal{N}(n)$. We show that $\Sigma^+ \subseteq [\![\bigwedge \Sigma]\!]$ and $\Sigma^- \subseteq [\![\neg \bigwedge \Sigma]\!]$, which implies $n \Vdash \Box \bigwedge \Sigma$. If $m \in \Sigma^+$, then $\mathsf{set}(\Sigma) \subseteq \Gamma_m$. By i.h. $m \Vdash A$ for all $A \in \Sigma$, then $m \Vdash \bigwedge \Sigma$.

If $m \in \Sigma^-$, then there is $B \in \Sigma \cap \Delta_m$. By i.h. $m \nVdash B$, then $m \nVdash \bigwedge \Sigma$. In the monotonic case the proof is analogous.

($\Box B \in \Gamma_n$) By saturation of $\mathsf{L}\Box$, $\langle B \rangle \in \Gamma_n$. Then by i.h. $n \Vdash \Box B$.

($\Box B \in \Delta_n$) In the non-monotonic case, assume $(\alpha, \beta) \in \mathcal{N}(n)$. Then there is $\langle \Sigma \rangle \in \Gamma_n$ such that $\Sigma^+ = \alpha$ and $\Sigma^- = \beta$. By saturation of rule $\mathsf{R}\Box$, there is $m \in \mathcal{W}$ such that $\Sigma \subseteq \Gamma_m$ and $B \in \Delta_m$, or there is $m \in \mathcal{W}$ such that $\Sigma \cap \Delta_m \neq \emptyset$ and $B \in \Gamma_m$. In the first case, $m \in \Sigma^+ = \alpha$ and by inductive hypothesis $m \nVdash B$, thus $\alpha \not\subseteq [\![B]\!]$. In the second case, $m \in \Sigma^- = \beta$ and by inductive hypothesis $m \Vdash B$, thus $\beta \cap [\![B]\!] \neq \emptyset$, i.e., $[\![B]\!] \not\subseteq \mathcal{W} \setminus \beta$. Therefore $n \nVdash \Box B$. The monotonic case is analogous.

Now we prove that if the failed proof is in $\mathbf{H.EX}^*$, then $\mathcal{M}$ satisfies condition (X).

(M) By definition, $\beta = \emptyset$ for every $(\alpha, \beta) \in \mathcal{N}(n)$.

(N) By saturation of rule $\mathsf{N}$, $\langle \top \rangle \in \Gamma_n$ for all $n \in \mathcal{W}$, thus $(\top^+, \top^-) \in \mathcal{N}(n)$. Moreover, by saturation of $\mathsf{R}\top$, $\top^- = \emptyset$.

(C) Assume that $(\alpha, \beta), (\gamma, \delta) \in \mathcal{N}(n)$. Then there are $\langle \Sigma \rangle, \langle \Pi \rangle \in \Gamma_n$ such that $\Sigma^+ = \alpha$, $\Sigma^- = \beta$, $\Pi^+ = \gamma$ and $\Pi^- = \delta$. By saturation or rule $\mathsf{C}$, there is $\langle \Omega \rangle \in \Gamma_n$ such that $\mathsf{set}(\Omega) = \mathsf{set}(\Sigma, \Pi)$, thus $(\Omega^+, \Omega^-) \in \mathcal{N}(n)$. We show that $(i)$ $\Omega^+ = \alpha \cap \gamma$ and $(ii)$ $\Omega^- = \beta \cup \delta$. $(i)$ $m \in \Omega^+$ iff $\mathsf{set}(\Omega) = \mathsf{set}(\Sigma, \Pi) \subseteq \Gamma_m$ iff $\mathsf{set}(\Sigma) \subseteq \Gamma_m$ and $\mathsf{set}(\Pi) \subseteq \Gamma_m$ iff $m \in \Sigma^+ = \alpha$ and $m \in \Pi^+ = \gamma$ iff $m \in \alpha \cap \gamma$. $(ii)$ $m \in \Omega^-$ iff $\Omega \cap \Delta_m \neq \emptyset$ iff $\Sigma, \Pi \cap \Delta_m \neq \emptyset$ iff $\Sigma \cap \Delta_m \neq \emptyset$ or $\Pi \cap \Delta_m \neq \emptyset$ iff $m \in \Sigma^- = \beta$ or $m \in \Pi^- = \delta$ iff $m \in \beta \cup \delta$.

(T) If $(\alpha, \beta) \in \mathcal{N}(n)$, then there is $\langle \Sigma \rangle \in \Gamma_n$ such that $\Sigma^+ = \alpha$ and $\Sigma^- = \beta$. By saturation of rule $\mathsf{T}$, $\mathsf{set}(\Sigma) \subseteq \Gamma_n$, then $n \in \Sigma^+ = \alpha$.

(P) If $(\alpha, \beta) \in \mathcal{N}(n)$, then there is $\langle \Sigma \rangle \in \Gamma_n$ such that $\Sigma^+ = \alpha$ and $\Sigma^- = \beta$. By saturation of rule $\mathsf{P}$, there is $m \in \mathcal{W}$ such that $\mathsf{set}(\Sigma) \subseteq \Gamma_m$, then $m \in \Sigma^+ = \alpha$, that is $\alpha \neq \emptyset$.

(D) Assume $(\alpha, \beta), (\gamma, \delta) \in \mathcal{N}(n)$. If $(\alpha, \beta) \neq (\gamma, \delta)$, then there are $\langle \Sigma \rangle, \langle \Pi \rangle \in \Gamma_n$ such that $\Sigma^+ = \alpha, \Sigma^- = \beta, \Pi^+ = \gamma$ and $\Pi^- = \delta$. If the calculus is non-monotonic, then by saturation of rule $\mathsf{D}_2$ there is $m \in \mathcal{W}$ such that $\mathsf{set}(\Sigma, \Pi) \subseteq \Gamma_m$ or there is $m \in \mathcal{W}$ such that $A, B \in \Delta_m$ for $A \in \Sigma$ and $B \in \Pi$. In the first case, $\mathsf{set}(\Sigma) \subseteq \Gamma_m$ and $\mathsf{set}(\Pi) \subseteq \Gamma_m$, thus $m \in \Sigma^+ = \alpha$ and $m \in \Pi^+ = \gamma$, that is $\alpha \cap \gamma \neq \emptyset$. In the second case, $m \in \Sigma^- = \beta$ and $m \in \Pi^- = \delta$, that is $\beta \cap \delta \neq \emptyset$. If in contrast the calculus is monotonic, by saturation of $\mathsf{D_M}$ there is $m \in \mathcal{W}$ such that $\mathsf{set}(\Sigma, \Pi) \subseteq \Gamma_m$. Then $\mathsf{set}(\Sigma) \subseteq \Gamma_m$ and $\mathsf{set}(\Pi) \subseteq \Gamma_m$, thus $m \in \Sigma^+ = \alpha$ and $m \in \Pi^+ = \gamma$, that is $\alpha \cap \gamma \neq \emptyset$. The other possibility is that $(\alpha, \beta) \neq (\gamma, \delta)$. Then there is $\langle \Sigma \rangle \in \Gamma_n$ such that $\Sigma^+ = \alpha$ and $\Sigma^- = \beta$. In the non-monotonic case, by saturation of $\mathsf{D}_1$ there is $m \in \mathcal{W}$ such that $\mathsf{set}(\Sigma) \subseteq \Gamma_m$ or there is $m \in \mathcal{W}$ such that $A \in \Delta_m$ for some $A \in \Sigma$. Then $m \in \Sigma^+ = \alpha$, that is $\alpha \neq \emptyset$, or $m \in= \Sigma^- = \beta$, that is $\beta \neq \emptyset$. In the monotonic case we can consider saturation of $\mathsf{P}$ and conclude that $\Sigma^+ = \alpha \neq \emptyset$.

($\mathrm{RD}_n^+$) Assume $(\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)$ be any $m \leq n$ different bi-neighbourhood pairs belonging to $\mathcal{N}(n)$. Then there are $\langle \Sigma_1 \rangle, \ldots, \langle \Sigma_m \rangle \in \Gamma_n$ such that $\Sigma_i^+ = \alpha_i$ and $\Sigma_i^- = \beta_i$ for every $1 \leq i \leq m$. By saturation of rule $\mathsf{D}_m^+$ (that by definition belongs to the calculus $\mathbf{H.ED_n^{+*}}$), there is $\ell \in \mathcal{W}$ such that $\mathsf{set}(\Sigma_1, \ldots, \Sigma_m) \subseteq \Gamma_\ell$. Then $\ell \in \Sigma_1^+ = \alpha_1, \ldots, \ell \in \Sigma_m^+ = \alpha_m$, that is $\alpha_1 \cap \ldots \cap \alpha_m \neq \emptyset$. $\Box$

Observe that, since all rules are cumulative, $\mathcal{M}$ is also a countermodel of the root hypersequent $H'$. Moreover, since every proof built in accordance with

the strategy either provides a derivation of the root hypersequent or contains a saturated hypersequent, this allows us to prove the following theorem.

**Theorem 6.3** (Semantic completeness)**.** If $H$ is valid in all bi-neighbourhood models for $\mathbf{E}^*$, then it is derivable in $\mathbf{H}.\mathbf{E}^*$.

*Proof.* Assume $H$ not derivable in $\mathbf{H}.\mathbf{E}^*$. Then there is a failed proof of $H$ in $\mathbf{H}.\mathbf{E}^*$ containing some saturated hypersequent $H'$. By Lemma 6.2, we can construct a bi-neighbourhood countermodel of $H'$, whence a countermodel of $H$, that satisfies all properties of bi-neighbourhood models for $\mathbf{E}^*$. Therefore $H$ is not valid in every bi-neighbourhood model for $\mathbf{E}^*$. □

Since the countermodels constructed for underivable hypersequents are based on the saturated hypersequents returned by Algorithm 1, and since the latter are finite, we immediately obtain the finite model property for all the logics. For the logics without $C$ we can further bound the *size* of the models, defined in the following way.

**Definition 6.3.** The *size* of a bi-neighbourhood or standard model $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$ is defined as $\mathsf{size}(\mathcal{M}) := |\mathcal{W}| + \sum_{w \in \mathcal{W}} |\mathcal{N}(w)|$.

**Corollary 6.4.** The logics without $C$ have the *polysize model property* wrt. bi-neighbourhood models, i.e., there is a polynomial $p$ such that if a formula $A$ of size $n$ is satisfiable, then it is satisfiable in a bi-neighbourhood model of size at most $p(n)$.

*Proof.* Given a underivable formula of size $n$, from the proof of Thm.5.4 we obtain that the saturated hypersequent used for constructing the countermodel has $\mathcal{O}(n^k)$ many components, each containing $\mathcal{O}(n)$ many blocks for $k$ depending only on the logic. Since the worlds of the countermodel correspond to the components, and the neighbourhoods for each world are constructed from the blocks occurring in that component, this model has at most $\mathcal{O}(n^k)$ many worlds, each with a neighbourhood of size at most $\mathcal{O}(n)$. Hence the size of the model is $\mathcal{O}(n^{k+1})$. □

As the above construction shows, we can directly extract a bi-neighbourhood countermodel from any failed proof. If we want to obtain a countermodel in the standard semantics we then need to apply the transformations presented in Section 3. In principle, the rough transformation (Proposition 3.4) can be embedded into the countermodel construction in order to directly construct a neighbourhood model, we just need to modify the definition of $\mathcal{N}(n)$ in Definition 6.2 as follows:

$$\mathcal{N}(n) = \{\gamma \mid \text{there is } \langle \Sigma \rangle \in \Gamma_n \text{ such that } \Sigma^+ \subseteq \gamma \subseteq \mathcal{W} \setminus \Sigma^-\}.$$

However, in this way we might obtain a model with a larger neighbourhood function than needed. In contrast, there is no obvious way to integrate the finer transformation of Proposition 3.5 into the countermodel construction, since it relies on the evaluation of formulas in an already existing model. But it does lead to smaller models:

**Corollary 6.5.** The logics without $C$ have the *polysize model property* wrt. standard models.

*Proof.* Given a satisfiable formula of size $n$, from Corollary 6.4 we obtain a bi-neighbourhood model with $\mathcal{O}(n)$ worlds. Since the transformation of Proposition 3.5 constructs neighbourhoods from sets of truth sets of subformulas of the input, the size of $\mathcal{N}_{st}(w)$ is at most $n$ for each world $w$. Hence the total size of the standard model is polynomial in the size $n$ of the formula. $\square$

An alternative way of obtaining countermodels in the standard neighbourhood semantics is proposed in [25]. It basically consists in forcing the proof search procedure to determine exactly the truth set of each formula. To this aim, whenever a sequent representing a new world is created, the sequent is saturated with respect to all disjunctions $A \vee \neg A$ such that $A$ is a subformula of the root sequent. This solution is equivalent to using *analytic cut* and makes the proof search procedure significantly more complex than the one given here.

Below we show some examples of countermodel extraction from failed proofs, both in the bi-neighbourhood and in the standard neighbourhood semantics. The latter are obtained by applying the transformation in Proposition 3.5.

**Example 6.1** (Proof search for axiom $M$ in **H.E** and countermodels)**.** The following is a failed proof of $\Box(p \wedge q) \Rightarrow \Box p$ in **H.E**.

$$\dfrac{\dfrac{\overset{\text{derivable}}{\langle p \wedge q\rangle, \Box(p \wedge q) \Rightarrow \Box p \mid p \wedge q \Rightarrow p} \qquad \dfrac{\overset{\text{derivable}}{\ldots \mid p \Rightarrow p \wedge q, p} \qquad \overset{\textbf{saturated}}{\langle p \wedge q\rangle, \Box(p \wedge q) \Rightarrow \Box p \mid p \Rightarrow p \wedge q, q}}{\langle p \wedge q\rangle, \Box(p \wedge q) \Rightarrow \Box p \mid p \Rightarrow p \wedge q}\,\mathsf{R}\wedge}{\langle p \wedge q\rangle, \Box(p \wedge q) \Rightarrow \Box p}\,\mathsf{R}\Box}{\Box(p \wedge q) \Rightarrow \Box p}\,\mathsf{L}\Box$$

<u>Bi-neighbourhood countermodel</u>. Let us consider the following enumeration of the compontents of the saturated hypersequent $H$: $1 \mapsto \langle p \wedge q\rangle, \Box(p \wedge q) \Rightarrow \Box p$; and $2 \mapsto p \Rightarrow p \wedge q, q$. According to the construction in Definition 6.2, from $H$ we obtain the following countermodel $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}_{bi}, \mathcal{V}\rangle$: $\mathcal{W} = \{1, 2\}$. $\mathcal{V}(p) = \{2\}$ and $\mathcal{V}(q) = \emptyset$. $\mathcal{N}_{bi}(2) = \emptyset$ and $\mathcal{N}_{bi}(1) = \{(\emptyset, \{2\})\}$, as $\mathcal{N}_{bi}(1) = \{(p \wedge q^+, p \wedge q^-)\}$ and $p \wedge q^+ = \emptyset$, $p \wedge q^- = \{2\}$. We have $1 \Vdash \Box(p \wedge q)$ because $\emptyset \subseteq [\![p \wedge q]\!] = \emptyset \subseteq \mathcal{W} \setminus \{2\}$, and $1 \nVdash \Box p$ because $[\![p]\!] = \{2\} \not\subseteq \mathcal{W} \setminus \{2\}$. Then $1 \nVdash \Box(p \wedge q) \to \Box p$.

<u>Neighbourhood countermodel</u>. We consider the set $\mathcal{S} = \{\Box(p \wedge q) \to \Box p, \Box(p \wedge q), \Box p, p \wedge q, p, q\}$ of the subformulas of $\Box(p \wedge q) \to \Box p$. By applying the transformation in Proposition 3.5 to the bi-neighbourhood model $\mathcal{M}_{bi}$, we obtain the standard model $\mathcal{M}_{st} = \langle \mathcal{W}, \mathcal{N}_{st}, \mathcal{V}\rangle$, where $\mathcal{W}$ and $\mathcal{V}$ are as in $\mathcal{M}_{bi}$, and $\mathcal{N}_{st}(1) = \{\emptyset\}$, since $\mathcal{N}_{st}(1) = \{[\![p \wedge q]\!]_{\mathcal{M}_{bi}}\}$ and $[\![p \wedge q]\!]_{\mathcal{M}_{bi}} = \emptyset$.

**Example 6.2** (Proof search for axiom $K$ in **H.EC** and countermodels)**.** If Figure 6 we find a failed proof of $\Box(p \to q) \to (\Box p \to \Box q)$ in **H.EC**. The countermodels are as follows.

<u>Bi-neighbourhood countermodel</u>. We consider the following enumeration of the compontents of the saturated hypersequent $H$:

$$\begin{array}{lcl} 1 & \mapsto & \Box(p \to q), \Box p, \langle p \to q\rangle, \langle p\rangle, \langle p \to q, p\rangle \Rightarrow \Box q. \\ 2 & \mapsto & q \Rightarrow p. \\ 3 & \mapsto & p \to q \Rightarrow q, p. \end{array}$$

According to the construction in Definition 6.2, from $H$ we obtain the following countermodel $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}_{bi}, \mathcal{V}\rangle$: $\mathcal{W} = \{1, 2, 3\}$. $\mathcal{V}(p) = \emptyset$ and $\mathcal{V}(q) = \{2\}$. $\mathcal{N}_{bi}(2) = \mathcal{N}_{bi}(3) = \emptyset$, and $\mathcal{N}_{bi}(1) = \{(\emptyset, \{2, 3\}), (\{3\}, \emptyset)\}$, as $\mathcal{N}_{bi}(1) =$

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\begin{array}{c}
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\end{array}
$$

**Figure 6:** Failed proof of axiom $K$ in **H.EC**.

$\{(p^+, p^-), (p \to q^+, p \to q^-), (p, p \to q^+, p, p \to q^-)\}$ and $p^+ = \emptyset$, $p^- = \{2, 3\}$, $p \to q^+ = \{3\}$, $p \to q^- = \emptyset$, $p, p \to q^+ = \emptyset$, $p, p \to q^- = \{2, 3\}$.

Then we have $1 \Vdash \Box(p \to q)$ because $\{3\} \subseteq [\![p \to q]\!] = \mathcal{W} \subseteq \mathcal{W} \setminus \emptyset$; and $x \Vdash \Box p$ because $\emptyset \subseteq [\![p]\!] = \emptyset \subseteq \mathcal{W} \setminus \{2, 3\}$; but $x \nVdash \Box q$ because $\{3\} \not\subseteq [\![q]\!] = \{2\}$ and $[\![q]\!] = \{2\} \not\subseteq \mathcal{W} \setminus \{2, 3\}$, whence $x \nVdash \Box(p \to q) \to (\Box p \to \Box q)$. Observe that $\mathcal{M}_{bi}$ is a C-model since $(\emptyset \cap \{3\}, \{2, 3\} \cup \emptyset) = (\emptyset, \{2, 3\})$.

Neighbourhood countermodel. By logical equivalence we can restrict the considered set of formulas $\mathcal{S}$ to $\{\Box(p \to q), \Box p, \Box q, p \to q, p, q, \Box((p \to q) \wedge q), \Box(p \wedge q)\}$. By the transformation in Proposition 3.5, from $\mathcal{M}_{bi}$ we obtain the standard model $\mathcal{M}_{st} = \langle \mathcal{W}, \mathcal{N}_{st}, \mathcal{V} \rangle$, where $\mathcal{W}$ and $\mathcal{V}$ are as in $\mathcal{M}_{bi}$, and $\mathcal{N}_{st}(1) = \{[\![p \to q]\!]_{\mathcal{M}_{bi}}, [\![p]\!]_{\mathcal{M}_{bi}}, [\![p \wedge q]\!]_{\mathcal{M}_{bi}}\} = \{\mathcal{W}, \emptyset\}$.

Finally, the next example shows the need of rule $\mathsf{D}_1$ for the calculus **H.ED** and its non-monotonic extensions from the point of view of the countermodel extraction.

**Example 6.3** (Proof search for $\neg\Box\top$ in **H.ED** and countermodel)**.** Let us consider the following failed proof of $\Box\top \Rightarrow$ in **H.ED**.

$$
\dfrac{\dfrac{\overset{\textbf{saturated}}{\Box\top, \langle\top\rangle \Rightarrow \mid \top \Rightarrow} \qquad \dfrac{}{\Box\top, \langle\top\rangle \Rightarrow \mid \Rightarrow \top}\,\mathsf{R}\top}{\Box\top, \langle\top\rangle \Rightarrow}\,\mathsf{D}_1}{\Box\top \Rightarrow}\,\mathsf{L}\Box
$$

Consider the saturated hypersequent and establish $1 \mapsto \Box\top, \langle\top\rangle \Rightarrow$, and $2 \mapsto \top \Rightarrow$. We obtain the bi-neighbourhood countermodel $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$, where $\mathcal{W} = \{1, 2\}$; $\mathcal{N}(1) = \{(\top^+, \top^-)\} = \{(\{2\}, \emptyset)\}$; and $\mathcal{N}(2) = \emptyset$. This is a D-model and falsifies $\neg\Box\top$, as $1 \Vdash \Box\top$.

Now imagine that the rule $\mathsf{D}_1$ does not belong to the calculus **H.ED**. In this case the proof would end with $\Box\top, \langle\top\rangle \Rightarrow$, as no other rule is backwards applicable to it. From this we would get the model $\mathcal{M}' = \langle \mathcal{W}', \mathcal{N}', \mathcal{V}' \rangle$, where $\mathcal{W}' = \{1\}$ and $\mathcal{N}'(1) = \{(\emptyset, \emptyset)\}$, which falsifies $\neg\Box\top$ but is *not* a D-model.

## Relational countermodels for regular logics

We now show that from failed proofs in $\mathbf{H.MC}^*$ it is also possible to directly extract relational countermodels of the non-derivable formulas (cf. Definition 3.4). This possibility not only makes the extraction of the relational models more efficient (as it prevents to go through the transformation of a previously extracted bi-neighbourhood model), but also shows the independency of the calculus from any specific semantic choice. Relational models are extracted from failed proofs in $\mathbf{H.MC}^*$ as follows.

**Definition 6.4** (Relational countermodel)**.** Let $H = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_k \Rightarrow \Delta_k$ be a saturated hypersequent occurring in a proof for $H'$ in $\mathbf{H.MC}^*$. For every $1 \leq n \leq k$, we say that a block $\langle\Sigma\rangle$ is *maximal* for $n$ if $\langle\Sigma\rangle \in \Gamma_n$, and for every $\langle\Pi\rangle \in \Gamma_n$, $\mathsf{set}(\Pi) \subseteq \mathsf{set}(\Sigma)$. It is easy to see that by saturation of rule $\mathsf{C}$ every component either contains a maximal block or does not contain any block at all. On the basis of $H$ we define the relational model $\mathcal{M} = \langle \mathcal{W}, \mathcal{W}^i, \mathcal{R}, \mathcal{V}\rangle$, as follows.

- $\mathcal{W}$, $\mathcal{V}$, and for every block $\langle\Sigma\rangle$, $\Sigma^+$, are defined as in Definition 6.2.
- $\mathcal{W}^i$ is the set of worlds $n$ such that $\Gamma_n$ does not contain any block.
- For every $n \in \mathcal{W} \setminus \mathcal{W}^i$, $\mathcal{R}(n) = \Sigma^+$, where $\langle\Sigma\rangle$ is a maximal block for $n$.

Observe that if $\langle\Sigma\rangle$ and $\langle\Pi\rangle$ are two maximal blocks for $n$, then $\Sigma^+ = \Pi^+$, whence $\mathcal{R}(n)$ is unique for every $n$.

**Lemma 6.6.** Let $H = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_k \Rightarrow \Delta_k$ be a saturated hypersequent occurring in a proof for $H'$ in $\mathbf{H.MC}^*$, and $\mathcal{M}$ be the model defined on the basis of $H$ as in Definition 6.4. Then for every formula $A$ and block $\langle\Sigma\rangle$ we have: if $A \in \Gamma_n$, then $\mathcal{M}, n \Vdash A$; if $\langle\Sigma\rangle \in \Gamma_n$, then $\mathcal{M}, n \Vdash \Box \bigwedge \Sigma$; and if $A \in \Delta_n$, then $\mathcal{M}, n \nVdash A$. Moreover, $\mathcal{M}$ is a relational model for $\mathbf{MC}$, and if $\mathbf{H.MC}^*$ contains rule $\mathsf{N}$, then $\mathcal{M}$ is a standard Kripke model for normal modal logic $\mathbf{K}$.

*Proof.* The truth lemma is proved by mutual induction on $A$ and $\langle\Sigma\rangle$. As usual we only consider modal formulas and blocks.

($\langle\Sigma\rangle \in \Gamma_n$) Then $n \in \mathcal{W} \setminus \mathcal{W}^i$. Moreover, given a block $\langle\Pi\rangle$ maximal for $n$, $\mathsf{set}(\Sigma) \subseteq \mathsf{set}(\Pi)$. We show that $\mathcal{R}(n) = \Pi^+ \subseteq [\![\bigwedge \Sigma]\!]$, which implies $n \Vdash \Box \bigwedge \Sigma$. If $m \in \Pi^+$, then $\mathsf{set}(\Pi) \subseteq \Gamma_m$, then for all $A \in \Pi$, $A \in \Gamma_m$, and by inductive hypothesis, $m \Vdash A$. Thus for all $A \in \Sigma$, $m \Vdash A$, that is $m \Vdash \bigwedge \Sigma$.

($A \in \Gamma_n$) By saturation of $\mathsf{L}\Box$, $\langle A\rangle \in \Gamma_n$. Then by inductive hypothesis, $n \Vdash \Box A$.

($A \in \Delta_n$) If there is no block in $\Gamma_n$, then $n \in \mathcal{W}^i$, and by definition $n \nVdash \Box A$. Otherwise, let $\langle\Sigma\rangle$ be a maximal block for $n$. Then by saturation of rule $\mathsf{R}\Box\mathsf{m}$ there is $m \in \mathcal{W}$ such that $\mathsf{set}(\Sigma) \subseteq \Gamma_m$ and $A \in \Delta_m$. Thus $m \in \Sigma^+ = \mathcal{R}(n)$, and by inductive hypothesis, $m \nVdash A$, therefore $\mathcal{R}(n) \not\subseteq [\![A]\!]$, which implies $n \nVdash \Box A$. $\Box$

As examples, we show failed proofs of axiom 4 in $\mathbf{H.MC}$ and $\mathbf{H.MCN}$ and the extracted countermodels.

**Example 6.4** (Proof search for axiom 4 in $\mathbf{H.MC}$ and countermodels)**.** A failed proof of 4 in $\mathbf{H.MC}$ is as follows.

$$\dfrac{\dfrac{\textbf{saturated}}{\Box p, \langle p\rangle \Rightarrow \Box\Box p \mid p \Rightarrow \Box p}\,\mathsf{R\Box m}}{\Box p, \langle p\rangle \Rightarrow \Box\Box p}$$
$$\dfrac{\Box p, \langle p\rangle \Rightarrow \Box\Box p}{\Box p \Rightarrow \Box\Box p}\,\mathsf{L\Box}$$

Let: $1 \mapsto \Box p, \langle p\rangle \Rightarrow \Box\Box p$; and $2 \mapsto p \Rightarrow \Box p$.

Bi-neighbourhood countermodel. From Definition 6.2 we obtain the following model $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V}\rangle$: $\mathcal{W} = \{1, 2\}$. $\mathcal{V}(p) = \{2\}$. $\mathcal{N}(1) = \{(p^+, p^-)\} = \{(\{2\}, \emptyset)\}$, and $\mathcal{N}(2) = \emptyset$. We have $\{2\} \subseteq [\![p]\!] = \{2\} \subseteq \mathcal{W} \setminus \emptyset$, then $1 \Vdash \Box p$, but $\{2\} \not\subseteq [\![\Box p]\!] = \{1\}$, then $1 \nVdash \Box\Box p$.

Relational countermodel. From Definition 6.4 we obtain the following model $\mathcal{M}_r = \langle \mathcal{W}, \mathcal{W}^i, \mathcal{R}, \mathcal{V}\rangle$: $\mathcal{W} = \{1, 2\}$ and $\mathcal{W}^i = \{2\}$. $\mathcal{V}(p) = \{2\}$. $\mathcal{R}(1) = p^+ = \{2\}$. Since $2 \Vdash p$ we have $1 \Vdash \Box p$. Moreover, since $2 \in \mathcal{W}^i$, by definition $2 \nVdash \Box p$, then $1 \nVdash \Box\Box p$.

**Example 6.5** (Proof search for axiom 4 in **H.MCNT** and countermodels)**.** A failed proof of 4 in **H.MCNT** is as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textbf{saturated}}{\Box p, \langle p\rangle, \langle\top\rangle, \langle p, \top\rangle, p, \top \Rightarrow \Box\Box p \mid p, \top, \langle\top\rangle \Rightarrow \Box p \mid \top, \langle\top\rangle \Rightarrow p}\,\mathsf{N}}{\Box p, \langle p\rangle, \langle\top\rangle, \langle p, \top\rangle, p, \top \Rightarrow \Box\Box p \mid p, \top, \langle\top\rangle \Rightarrow \Box p \mid \top \Rightarrow p}\,\mathsf{R\Box m}}{\Box p, \langle p\rangle, \langle\top\rangle, \langle p, \top\rangle, p, \top \Rightarrow \Box\Box p \mid p, \top, \langle\top\rangle \Rightarrow \Box p}\,\mathsf{N}}{\Box p, \langle p\rangle, \langle\top\rangle, \langle p, \top\rangle, p, \top \Rightarrow \Box\Box p \mid p, \top \Rightarrow \Box p}\,\mathsf{R\Box m}}{\Box p, \langle p\rangle, \langle\top\rangle, \langle p, \top\rangle, p, \top \Rightarrow \Box\Box p}\,\mathsf{T}}{\Box p, \langle p\rangle, \langle\top\rangle, \langle p, \top\rangle \Rightarrow \Box\Box p}\,\mathsf{C}}{\Box p, \langle p\rangle, \langle\top\rangle \Rightarrow \Box\Box p}\,\mathsf{N}}{\Box p, \langle p\rangle \Rightarrow \Box\Box p}$$
$$\dfrac{\Box p, \langle p\rangle \Rightarrow \Box\Box p}{\Box p \Rightarrow \Box\Box p}\,\mathsf{L\Box}$$

Let: $1 \mapsto \Box p, \langle p\rangle, \langle\top\rangle, \langle p, \top\rangle, p, \top \Rightarrow \Box\Box p$; $2 \mapsto p, \top, \langle\top\rangle \Rightarrow \Box p$; and $3 \mapsto \top, \langle\top\rangle \Rightarrow p$.

Bi-neighbourhood countermodel. From Definition 6.2 we obtain the following model $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V}\rangle$: $\mathcal{W} = \{1, 2, 3\}$. $\mathcal{V}(p) = \{1, 2\}$.
$\mathcal{N}(1) = \{(p^+, p^-), (\top^+, \top^-), (p, \top^+; p, \top^-)\} = \{(\{1, 2\}, \{3\}), (\{1, 2, 3\}, \emptyset)\}$.
$\mathcal{N}(2) = \mathcal{N}(3) = \{(\top^+, \top^-)\} = \{(\{1, 2, 3\}, \emptyset)\}$. It is easy to see that $\mathcal{M}_{bi}$ is a MCNT-model. Moreover, $1 \Vdash \Box p$ and $1 \nVdash \Box\Box p$, then $1 \nVdash \Box p \to \Box\Box p$.

Relational countermodel. From Definition 6.4 we obtain the following model $\mathcal{M}_r = \langle \mathcal{W}, \mathcal{W}^i, \mathcal{R}, \mathcal{V}\rangle$: $\mathcal{W} = \{1, 2, 3\}$ and $\mathcal{W}^i = \emptyset$. $\mathcal{V}(p) = \{1, 2\}$. $\mathcal{R}(1) = (p, \top)^+ = \{1, 2\}$; and $\mathcal{R}(2) = \mathcal{R}(3) = \top^+ = \{1, 2, 3\}$. Since $1 \Vdash p$ and $2 \Vdash p$, $1 \Vdash \Box p$. But $3 \nVdash p$, then $2 \nVdash \Box p$, thus $1 \nVdash \Box\Box p$. Then we have $1 \nVdash \Box p \to \Box\Box p$. Notice that $\mathcal{R}$ is reflexive but is not transitive, as $1\mathcal{R}2$, $2\mathcal{R}3$, but not $1\mathcal{R}3$.

# 7 On translations for the classical cube

Different proof-theoretical frameworks can be used for specifying axiomatic systems, and there are many possible reasons for preferring one over the other. The best known (and maybe simplest) formalism for analytic proof systems is Gentzen's *sequent calculus* [14]. But simplicity often implies less comprehensiveness, and here it is not different: although being an ideal tool for proving

meta-logical properties, sequent calculus is not expressive enough for constructing analytic calculi for many logics of interest. Moreover, sequent rules seldom reflect the semantic characterisation of the logic. As a result, many new formalisms have been proposed over the last 30 years, including hypersequent calculi, but also *labelled calculi* [41]. Hypersequents and labelled sequents are very different in nature since, in the latter, the basic objects are usually formulas of a more expressive language reflecting the logic's semantics. Hypersequent systems, in contrast, are generalisations of sequent systems, carrying a more syntactic characteristic, and are sometimes considered an *antagonist* formalism with respect to labelled calculi [6].

In the present work, we showed how hypersequents adequately reflect the semantics of bi-neighbourhood models. In [9] the labelled sequent calculi $\mathbf{LSE}^*$, were developed for all the logics of the classical cube. These calculi also reflect the bi-neighbourhood semantics, and are fully modular. In this section, we will show that, in the case of NNMLs, hypersequents and labels are far from being antagonists. In fact, we show that they are strongly related, by presenting translations between $\mathbf{H.E}^*$ (Figure 4) restricted to the classical cube and the labelled calculi $\mathbf{LSE}^*$, presented next.

The language $\mathcal{L}_{\mathsf{LS}}$ of labelled calculi extends $\mathcal{L}$ with a set $WL = \{x, y, z, ...\}$ of *world labels*, and a set $NL = \{a, b, c, ...\}$ of *neighbourhood labels*. We define *positive neighbourhood terms*, written $a_1 \dots a_n$, as finite multisets of neighbourhood labels. Moreover, if $t$ is a positive term, then $\bar{t}$ is a negative term. Negative terms $\bar{t}$ cannot be proper subterms, in particular cannot be negated. The term $\tau$ and its negative counterpart $\overline{\tau}$ are neighbourhood constants. We will represent by $\mathsf{t}$ either $t$ or $\bar{t}$.

Intuitively, positive (resp. negative) terms represent the intersection (resp. the union) of their constituents. Moreover, $t$ and $\bar{t}$ are the two members of a pair of neighbourhoods in bi-neighbourhood models. Observe that the operation of overlining a term cannot be iterated: it can be applied only once for turning a positive term into a negative one. The operations of composition and substitution over positive terms are defined as usual (see [9]).

The formulas of $\mathcal{L}_{\mathsf{LS}}$ are of the following kinds and respective intuitive interpretation

| | | | |
|---|---|---|---|
| $\phi$ | $::=$ | $x : A$ | $A$ is satisfied by $x$ |
| | | $\mid t \Vdash^{\forall} A$ | $A$ is satisfied by every world in the neighbourhood $t$ |
| | | $\mid \bar{t} \Vdash^{\exists} A$ | $A$ is satisfied by some world in the neighbourhood $\bar{t}$ |
| | | $\mid x \in \mathsf{t}$ | $x$ is a world in the neighbourhood $\mathsf{t}$ |
| | | $\mid t \rhd x$ | the pair $(t, \bar{t})$ is a bi-neighbourhood of $x$. |

*Sequents* are pairs $\Gamma \Rightarrow \Delta$ of multisets of formulas of $\mathcal{L}_{\mathsf{LS}}$. The fully modular calculi $\mathbf{LSE}^*$ are defined by the rules in Figure 7.

We are interested in the translation of *derivations* between $\mathbf{H.E}^*$ and $\mathbf{LSE}^*$. We start by explaining some choices made thorough this work.

**Hypersequents.** As already shown, hypersequents present an elegant and modular solution for addressing non-normal modalities. This is mainly due to two facts: (1) negative occurrences of boxed formulas are organized into blocks; and (2) components are independent once created. Hence proof search avoids the non-determinism often generated by component *communication* rules [2],

**Propositional rules**

$$\mathsf{init}\ \frac{}{x : p, \Gamma \Rightarrow \Delta, x : p} \qquad \mathsf{L}\bot\ \frac{}{x : \bot, \Gamma \Rightarrow \Delta} \qquad \mathsf{R}\top\ \frac{}{\Gamma \Rightarrow \Delta, x : \top}$$

$$\mathsf{L}\to\ \frac{\Gamma \Rightarrow \Delta, x : A \qquad x : B, \Gamma \Rightarrow \Delta}{x : A \to B, \Gamma \Rightarrow \Delta} \qquad \mathsf{R}\to\ \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \to B}$$

$$\mathsf{L}\wedge\ \frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A \wedge B, \Gamma \Rightarrow \Delta} \qquad \mathsf{R}\wedge\ \frac{\Gamma \Rightarrow \Delta, x : A \qquad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \wedge B}$$

$$\mathsf{L}\vee\ \frac{x : A, \Gamma \Rightarrow \Delta \qquad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta} \qquad \mathsf{R}\vee\ \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}$$

**Rules for the classical cube**

$$\mathsf{L}\Box\ \frac{a \rhd x, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} A}{x : \Box A, \Gamma \Rightarrow \Delta}\ (a!)$$

$$\mathsf{R}\Box\ \frac{t \rhd x, \Gamma \Rightarrow \Delta, x : \Box A, t \Vdash^{\forall} A \qquad t \rhd x, \overline{t} \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x : \Box A}{t \rhd x, \Gamma \Rightarrow \Delta, x : \Box A}$$

$$\mathsf{M}\ \frac{}{t \rhd x, y \in \overline{t}, \Gamma \Rightarrow \Delta} \qquad \mathsf{N}\ \frac{\tau \rhd x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (x \text{ in } \Gamma \cup \Delta) \qquad \mathsf{C}\ \frac{ts \rhd x, t \rhd x, s \rhd x, \Gamma \Rightarrow \Delta}{t \rhd x, s \rhd x, \Gamma \Rightarrow \Delta}$$

**Rules for local forcing**

$$\mathsf{L} \Vdash^{\forall}\ \frac{x \in t, x : A, t \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x \in t, t \Vdash^{\forall} A, \Gamma \Rightarrow \Delta} \qquad \mathsf{R} \Vdash^{\forall}\ \frac{y \in t, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, t \Vdash^{\forall} A}\ (y!)$$

$$\mathsf{L} \Vdash^{\exists}\ \frac{y \in \overline{t}, y : A, \Gamma \Rightarrow \Delta}{\overline{t} \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}\ (y!) \qquad \mathsf{R} \Vdash^{\exists}\ \frac{x \in \overline{t}, \Gamma \Rightarrow \Delta, x : A, \overline{t} \Vdash^{\exists} A}{x \in \overline{t}, \Gamma \Rightarrow \Delta, \overline{t} \Vdash^{\exists} A}$$

**Rules for neighbourhood terms**

$$\mathsf{dec}\ \frac{x \in t, x \in s, x \in ts, \Gamma \Rightarrow \Delta}{x \in ts, \Gamma \Rightarrow \Delta} \qquad \overline{\mathsf{dec}}\ \frac{x \in \overline{t}, x \in \overline{ts}, \Gamma \Rightarrow \Delta \qquad x \in \overline{s}, x \in \overline{ts}, \Gamma \Rightarrow \Delta}{x \in \overline{ts}, \Gamma \Rightarrow \Delta}$$

$$\overline{\tau}^{\emptyset}\ \frac{}{x \in \overline{\tau}, \Gamma \Rightarrow \Delta}$$

Application conditions:
$y$ is fresh in $\mathsf{R} \Vdash^{\forall}$and $\mathsf{L} \Vdash^{\exists}$, $a$ is fresh in $\mathsf{L}\Box$, and $x$ occurs in the conclusion of $\mathsf{N}$.

**Figure 7:** Rules of labelled sequent calculi **LSE**$^*$.

establishing a straight-forward proof-search procedure. This is reflected in the left and right rules for the $\Box$

$$\mathsf{L}\Box\ \frac{G \mid \Gamma, \Box A, \langle A \rangle \Rightarrow \Delta}{G \mid \Gamma, \Box A \Rightarrow \Delta} \qquad \mathsf{R}\Box\ \frac{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid \Sigma \Rightarrow B \qquad \{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid B \Rightarrow A\}_{A \in \Sigma}}{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta}$$

Reading rules from the conclusion upwards, the $\mathsf{L}\Box$ rule substitutes a $\Box$ with a block. Blocks can then gather more formulas only by the applications of the $\mathsf{C}$ rule. Applications of the rule $\mathsf{R}\Box$ closes this proof cycle, creating new components involving only right-boxed and blocked formulas, and immediately closing the communication between components.

This determines a proof search procedure, where propositional rules can be eagerly applied until only blocks remain and a non-deterministic choice is triggered, where blocs/boxed formulas should be combined for producing new components. The invertibility of rules attenuates such non-determinism: allowing the generation of all possible combinations avoids the need for backtracking.

But not only that: our calculi are greatly inspired and supported by the choice of the *semantics*.

**Bi-neighbourhood.** As pointed out in the introduction, in the bi-neighbourhood semantics the elements of a pair provide positive and negative support for a modal formula. This is fully captured by the box rules: the $\mathsf{L}\Box$ rule places formulas into fresh neighbourhoods, the rule $\mathsf{C}$ joins such formulas into intersections of neighbourhoods and the $\mathsf{R}\Box$ rule carries the formulas of a chosen neighbourhood together with a right-boxed formula into a fresh world belonging to this neighbourhood.

These ideas can be also interpreted using *labels*.

**Labels.** The labelled counterparts for the box rules are

$$\mathsf{L}\Box\ \frac{a \rhd x, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} A}{x : \Box A, \Gamma \Rightarrow \Delta} \qquad \mathsf{R}\Box\ \frac{t \rhd x, \Gamma \Rightarrow \Delta, x : \Box A, t \Vdash^{\forall} A \qquad t \rhd x, \overline{t} \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x : \Box A}{t \rhd x, \Gamma \Rightarrow \Delta, x : \Box A}$$

Starting from a labelled sequent $S$ placed in a component labelled by a world-variable $x_1$, the $\mathsf{L}\Box$ rule over $\Box A^1_{ij}$ creates a fresh neighbourhood-variable $a^1_{ij}$ of $x_1$, placing $A^1_{ij}$ in it. The rule $\mathsf{C}$ then joins formulas $A^1_{ij}, j = 1, \ldots, s_{i1}$ into blocks $\langle \Sigma^1_i \rangle, i = 1, \ldots, l_1$, given by the intersection of the neighbourhoods $a^1_{ij}$, represented by $a^1_i = a^1_{i1} \ldots a^1_{is_{i1}}$. That is, the blocks $\langle \Sigma^1_i \rangle$ formed from $S$ carry the information of boxed formulas, grouped into subsets $\{a^1_i\}_i$, determined by neighbourhood intersections.

The $\mathsf{R}\Box$ rule then non-deterministically choses one of such blocks $a^1_i$ for $i \in \{1, \ldots l_1\}$ and a right-boxed formula $B$, creating a fresh world-variable $x_2$ in $a^1_i$ and placing $B$ and $A^1_{ij}$, for all $j = 1, \ldots, s_{i1}$, under this world. Observe that the left and right premises of rule $\mathsf{R}\Box$ reflect the positive and negative support for the modal formula $B$.

This strongly highlights the similarities between the hyper and label formalisms. We define next a translation from hypersequents to labelled sequents.

**Definition 7.1.** Let $\Sigma^k_i = \{A^k_{ij}\}$, $i = 1, \ldots, l_k; j = 1, \ldots, s_{ik}; k = 1, \ldots n$ and fix a hypersequent enumeration (see Definition 6.2). The translation $[\cdot]^{\mathsf{a}_n}_{\mathsf{x}_n}$ from the hypersequent to the labelled languages, parametric on the world and neighbourhood labels $\mathsf{a}_n$ and $\mathsf{x}_n$, respectively, is recursively defined as

$$
\begin{array}{lcl}
\left[\Gamma, \langle \Sigma_i^1 \rangle \Rightarrow \Delta\right]_{\mathsf{x}_1}^{\mathsf{a}_1} & = & \{a_i^1 \rhd x_1\}_i, \{a_{ij}^1 \Vdash^{\forall} A_{ij}^1\}_{ij}, x_1 : \Gamma \Rightarrow x_1 : \Delta, \{\overline{a_{ij}^1} \Vdash^{\exists} A_{ij}^1\}_{ij} \\
[G \mid \Gamma, \langle \Sigma_i^n \rangle \Rightarrow \Delta]_{\mathsf{x}_n}^{\mathsf{a}_n} & = & [G]_{\mathsf{x}_{n-1}}^{\mathsf{a}_{n-1}} \otimes (x_n \in b_n, \{a_i^n \rhd x_n\}_i, \{a_{ij}^n \Vdash^{\forall} A_{ij}^n\}_{ij}, x_n : \Gamma \Rightarrow \\
 & & x_n : \Delta, \{\overline{a_{ij}^n} \Vdash^{\exists} A_{ij}^n\}_{ij})
\end{array}
$$

where

- $k = 1, \ldots, n$ indexes the components;
- $i = 1, \ldots, l_k$ indexes the blocks in the component $k$;
- $j = 1, \ldots, s_{ik}$ indexes the formulas in the block $i$ of the component $k$;
- $\mathsf{x}_n = \{x_k\}_{1 \leq k \leq n}$, where $x_k$ is a world variable relative to the $k$-th component;
- $\mathsf{a}_n = \bigcup_{k=1}^{n} \{a_i^k\}_{1 \leq i \leq l_k}$, where $\{a_i^k\}_{1 \leq i \leq l_k}$ is the set of neighbourhood variables representing blocks in the $k$-th component, $a_i^k = a_{i1}^k \ldots a_{is_{ik}}^k$;
- $b_n \in \{a_i^k\}_i \cup \{\overline{a_i^k}\}_i$ is a neighbourhood term, with $1 \leq k \leq n-1$;
- the operator $\otimes$ represents the concatenation of sequents

$$(\Theta_1 \Rightarrow \Upsilon_1) \otimes (\Theta_2 \Rightarrow \Upsilon_2) := (\Theta_1, \Theta_2 \Rightarrow \Upsilon_1, \Upsilon_2)$$

For readability, we will ease the notation by assuming that: the active component in the conclusion of rule applications has label $n$; $\langle \Sigma \rangle$ is a block in this component with $\Sigma = \{A_j\}, 1 \leq j \leq s$; and $a = a_1 \ldots a_s$ is the neighbourhood variable representing $\langle \Sigma \rangle$, where $a = \tau$ if $\Sigma = \{\top\}$; if $b$ is the neighbourhood variable representing $\langle \Pi \rangle$, then $ab$ is the neighbourhood variable representing $\langle \Sigma, \Pi \rangle$. Finally, we will omit the non-active formulas on the derivations, replaced by (possibly indexed) context variables $X, Y$.

Observe that hypersequent and labelled proofs have two important differences: rules in $\mathbf{H.E}^*$ are *Kleene'd*, in the sense that principal formulas are explicitly copied bottom-up; and $\mathbf{LSE}^*$ introduces terms and proof-steps that have no correspondence in the hypersequent setting. As a result, we need to introduce some flexibility in how contexts are related between a hypersequent proof and the labelled proof emulating it.

Let $R$ be either a propositional or a left box rule in $\mathbf{H.E}^*$, and $H$ one of its premises. We say that $U_H$ is an *un-Kleene'd* version of $H$ if $U_H$ coincides with $H$ but for the replication of the principal formula, in an application of $R$ (see also Section 5). For example, in the derivation

$$\frac{H = G \mid \Gamma, \Box A, \langle A \rangle \Rightarrow \Delta}{G \mid \Gamma, \Box A \Rightarrow \Delta}$$

we have that $U_H = G \mid \Gamma, \langle A \rangle \Rightarrow \Delta$.

Similarly, consider an application of the $\mathsf{R}\Box$ rule in $\mathbf{H.E}^*$, with conclusion $H$ and premises $H_1, H_2^j, 1 \leq j \leq s$

$$\frac{H_1 = G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid \Sigma \Rightarrow B \quad \{H_2^j = G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid B \Rightarrow A_j\}_{A_j \in \Sigma}}{H = G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta} \ \mathsf{R}\Box$$

and an application of the $\mathsf{R}\Box$ rule in $\mathbf{LSE}^*$, with conclusion $S = [H]^{\mathsf{a}_n}_{\mathsf{x}_n}$ and premises $S_1, S_2$

$$\frac{S_1 = [H]^{\mathsf{a}_n}_{\mathsf{x}_n} \otimes (\Rightarrow a \Vdash^{\forall} B) \quad S_2 = [H]^{\mathsf{a}_n}_{\mathsf{x}_n} \otimes (\overline{a} \Vdash^{\exists} B \Rightarrow)}{S = [H]^{\mathsf{a}_n}_{\mathsf{x}_n}} \ \mathsf{R}\Box$$

Let $x_{n+1}$ be a fresh world variable. We call

$E_{H_1} = [H_1]^{\mathsf{a}_n}_{\mathsf{x}_{n+1}} \otimes (x_{n+1} \in a, \{x_{n+1} \in a_j\}_j \Rightarrow)$ and $E_{H_2^j} = \left[H_2^j\right]^{\mathsf{a}_n}_{\mathsf{x}_{n+1}} \otimes (x_{n+1} \in \overline{a}, x_{n+1} \in \overline{a_j} \Rightarrow)$ *extensions* of $[H_1]^{\mathsf{a}_n}_{\mathsf{x}_{n+1}}$ and $\left[H_2^j\right]^{\mathsf{a}_n}_{\mathsf{x}_{n+1}}$, respectively.[2]

The following lemma shows that unkleening and extensions do not alter provability.

**Lemma 7.1.** Let $H, U_H, S_1, S_2, E_{H_1}, E_{H_2^j}$ as described above. Then

a. $H$ and $U_H$ are height-preserving equivalent in $\mathbf{H.E}^*$, that is, $H$ is provable with height at most $n$ in $\mathbf{H.E}^*$ iff so it is $U_H$;

b. $S_1$ (resp. $S_2$) is provable iff $E_{H_1}$ is provable (resp. $E_{H_2^j}$ is provable, for all $1 \leq j \leq s$) in $\mathbf{LSE}^*$;

c. $E_{H_1}$ and $[H_1]^{\mathsf{a}_n}_{\mathsf{x}_{n+1}}$ (resp. $E_{H_2^j}$ and $\left[H_2^j\right]^{\mathsf{a}_n}_{\mathsf{x}_{n+1}}$) are height-preserving equivalent in $\mathbf{LSE}^*$.

*Proof.* (a) is easily proven by the usual invertibility argument. Regarding (b), observe that all the rules in $\mathbf{LSE}^*$ are invertible. Hence, in the derivation $\pi_1$:

$$\cfrac{\cfrac{\cfrac{E_{H_1} = X, \{a_j \Vdash^{\forall} A_j\}_j, \{x_{n+1} \in a_j\}_j, x_{n+1} \in a, \{x_{n+1} : A_j\}_j \Rightarrow x_{n+1} : B, Y}{X, \{a_j \Vdash^{\forall} A_j\}_j, \{x_{n+1} \in a_j\}_j, x_{n+1} \in a \Rightarrow x_{n+1} : B, Y}\ \mathsf{L}\Vdash^{\forall}}{X, \{a_j \Vdash^{\forall} A_j\}_j, x_{n+1} \in a \Rightarrow x_{n+1} : B, Y}\ \mathsf{dec}}{S_1 = X, \{a_j \Vdash^{\forall} A_j\}_j \Rightarrow a \Vdash^{\forall} B, Y}\ \mathsf{R}\Vdash^{\forall}$$

the sequent $S_1$ is provable iff $E_{H_1}$ is provable. Analogously for the derivation $\pi_2$:

$$\cfrac{\cfrac{\cfrac{\{E_{H_2^j} = X, x_{n+1} \in \overline{a}, x_{n+1} \in \overline{a_j}, x_{n+1} : B \Rightarrow x_{n+1} : A_j, \{\overline{a_j} \Vdash^{\exists} A_j\}_j, Y\}_j}{\{X, x_{n+1} \in \overline{a}, x_{n+1} \in \overline{a_j}, x_{n+1} : B \Rightarrow \{\overline{a_j} \Vdash^{\exists} A_j\}_j, Y\}_j}\ \mathsf{R}\Vdash^{\exists}}{X, x_{n+1} \in \overline{a}, x_{n+1} : B \Rightarrow \{\overline{a_j} \Vdash^{\exists} A_j\}_j, Y}\ \overline{\mathsf{dec}}}{S_2 = X, \overline{a} \Vdash^{\exists} B \Rightarrow \{\overline{a_j} \Vdash^{\exists} A_j\}_j, Y}\ \mathsf{L}\Vdash^{\exists}$$

Finally, for (c), assume that there is a proof $\pi$ of $E_{H_1}$ with height $n$. Observe that the only rules that can be applied over $x_{n+1} \in a$ and $x_{n+1} \in a_j$ in $\pi$ are $\mathsf{dec}$ and $\mathsf{L}\Vdash^{\forall}$, respectively. But applying such rules would only duplicate formulas already in $E_{H_1}$, and thus could be eliminated. Hence $\pi$ can be transformed into a proof $\pi'$ of $E_{H_1}$ with height at most $n$, where no rules are applied over $x_{n+1} \in a$ or $x_{n+1} \in a_j$, and the result follows. The case for $E_{H_2^j}$ is similar. $\Box$

The next result establishes the relationship between $\mathbf{H.E}^*$ and $\mathbf{LSE}^*$.

[2] Here we slightly abuse the notation since $\mathsf{a}_{n+1} = \mathsf{a}_n$.

**Theorem 7.2.** Let $H$ be a hypersequent in $\mathbf{H.E}^*$ with length $n$. The following are equivalent.

1. $H$ is provable in $\mathbf{H.E}^*$;
2. $[H]^{\mathsf{a}_n}_{\mathsf{x}_n}$ is provable in $\mathbf{LSE}^*$.

*Proof.* Consider the following translation between hypersequent *rule applications* and *derivations* in the labelled calculi, where the translation for the propositional rules is the trivial one.

- Case $\mathsf{L}\Box$.

$$\frac{H = G \mid \Gamma, \Box A, \langle A \rangle \Rightarrow \Delta}{G \mid \Gamma, \Box A \Rightarrow \Delta} \quad \rightsquigarrow \quad \frac{[U_H = G \mid \Gamma, \langle A \rangle \Rightarrow \Delta]^{\mathsf{a}_n \cup \{a\}}_{\mathsf{x}_n}}{[G \mid \Gamma, \Box A \Rightarrow \Delta]^{\mathsf{a}_n}_{\mathsf{x}_n}} \ \mathsf{L}\Box$$

where $a$ is a fresh neighbourhood variable to be added, in $\mathsf{a}_n$, to the set of neighbourhood variables representing blocks in the $n$-th component.

- Case $\mathsf{R}\Box$.

$$\frac{H_1 = G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid \Sigma \Rightarrow B \quad \{H_2^j = G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid B \Rightarrow A_j\}_j}{H = G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta}$$

$\rightsquigarrow$

$$\dfrac{\begin{array}{c} E_{H_1} = [H_1]^{\mathsf{a}_n}_{\mathsf{x}_{n+1}} \otimes (x_{n+1} \in a, \{x_{n+1} \in a_j\}_j \Rightarrow ) \\ \vdots\, \pi_1 \\ S_1 = [H]^{\mathsf{a}_n}_{\mathsf{x}_n} \otimes ( \Rightarrow a \Vdash^{\forall} B) \end{array} \quad \begin{array}{c} \{E_{H_2^j} = \left[H_2^j\right]^{\mathsf{a}_n}_{\mathsf{x}_{n+1}} \otimes (x_{n+1} \in \overline{a}, x_{n+1} \in \overline{a_j} \Rightarrow )\}_j \\ \vdots\, \pi_2 \\ S_2 = [H]^{\mathsf{a}_n}_{\mathsf{x}_n} \otimes (\overline{a} \Vdash^{\exists} B \Rightarrow ) \end{array}}{[H]^{\mathsf{a}_n}_{\mathsf{x}_n}} \ \mathsf{R}\Box$$

where $\pi_1, \pi_2$ are the derivations in the proof Lemma 7.1 (b).

- Case $\mathsf{M}$. Similar and simpler to the case $\mathsf{R}\Box$, since the right premise in the derivation above has the proof

$$\dfrac{\dfrac{}{X, a \rhd x, x_{n+1} \in \overline{a}, x_{n+1} : B \Rightarrow Y}\ \mathsf{M}}{X, a \rhd x, \overline{a} \Vdash^{\exists} B \Rightarrow Y}\ \mathsf{L} \Vdash^{\exists}$$

Hence,

$$\frac{H_1 = G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid \Sigma \Rightarrow B}{H = G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta} \quad \rightsquigarrow \quad \dfrac{\begin{array}{c} E_{H_1} = [H_1]^{\mathsf{a}_n}_{\mathsf{x}_{n+1}} \otimes (x_{n+1} \in a, \{x_{n+1} \in a_j\}_j \Rightarrow ) \\ \vdots\, \pi_1 \\ S_1 = [H]^{\mathsf{a}_n}_{\mathsf{x}_n} \otimes ( \Rightarrow a \Vdash^{\forall} B) \end{array}}{[H]^{\mathsf{a}_n}_{\mathsf{x}_{n+1}}} \ \mathsf{R}\Box$$

- Case $\mathsf{C}$.

$$\frac{G \mid \Gamma, \langle \Sigma \rangle, \langle \Pi \rangle, \langle \Sigma, \Pi \rangle \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma \rangle, \langle \Pi \rangle \Rightarrow \Delta} \quad \rightsquigarrow \quad \frac{[G \mid \Gamma, \langle \Sigma \rangle, \langle \Pi \rangle, \langle \Sigma, \Pi \rangle \Rightarrow \Delta]^{\mathsf{a}_n}_{\mathsf{x}_n}}{[G \mid \Gamma, \langle \Sigma \rangle, \langle \Pi \rangle \Rightarrow \Delta]^{\mathsf{a}_n}_{\mathsf{x}_n}} \ \mathsf{C}$$

- Case $\mathsf{N}$.

$$\frac{G \mid \Gamma, \langle\top\rangle \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta} \quad \rightsquigarrow \quad \frac{[G \mid \Gamma \Rightarrow \Delta]^{\mathsf{a}_n}_{\mathsf{x}_n} \otimes (\tau \rhd x_n \Rightarrow\ )}{[G \mid \Gamma \Rightarrow \Delta]^{\mathsf{a}_n}_{\mathsf{x}_n}}\ \mathsf{N}$$

Observe that $[G \mid \Gamma, \langle\top\rangle \Rightarrow \Delta]^{\mathsf{a}_n}_{\mathsf{x}_n} = [G \mid \Gamma \Rightarrow \Delta]^{\mathsf{a}_n}_{\mathsf{x}_n} \otimes (\tau \rhd x_n, \tau \Vdash^{\forall} \top \Rightarrow \overline{\tau} \Vdash^{\exists} \top)$. But this sequent is provable iff $[G \mid \Gamma \Rightarrow \Delta]^{\mathsf{a}_n}_{\mathsf{x}_n} \otimes (\tau \rhd x_n \Rightarrow\ )$ is provable, since $\tau \Vdash^{\forall} \top$ can only add $x : \top$ to the right context, while $\overline{\tau} \Vdash^{\exists} \top$ can only be triggered if $x \in \overline{\tau}$ is already in the left context for some $x$.

Given this transformation and in the view of Lemma 7.1, (1) $\Rightarrow$ (2) is easily proved by induction on a proof of $H$ in $\mathbf{H}.\mathbf{E}^*$.

For proving (2) $\Rightarrow$ (1) observe that *provability* is maintained from the end-sequent to the open leaves in the translated derivations. This means that choosing a formula $[H]^{\mathsf{a}_n}_{\mathsf{x}_n}$ to work on is equivalent to performing all the steps of the translation given above, ending with translated hypersequents of smaller proofs. This is, in fact, one of the pillars of the *focusing* method [28].

In order to illustrate this, let $H = G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Box B, \Delta$ and consider the following derivation in the monotonic case

$$\frac{[H]^{\mathsf{a}_n}_{\mathsf{x}_n} \otimes (\overset{\pi}{\Rightarrow} a \Vdash^{\forall} B)}{[H]^{\mathsf{a}_n}_{\mathsf{x}_n}}\ \mathsf{R}\Box$$

where one decides to work on $[H]^{\mathsf{a}_n}_{\mathsf{x}_n}$. If $a \Vdash^{\forall} B$ is never principal in $\pi$, then $\pi$ acts over $[H]^{\mathsf{a}_n}_{\mathsf{x}_n}$ only and this derivation can be substituted by

$$\frac{\begin{array}{c}[H]^{\mathsf{a}_n}_{\mathsf{x}_n} \otimes (x_{n+1} \in a, \{x_{n+1} \in a_j\}_j, \{x_{n+1} : A_j\}_j \Rightarrow x_{n+1} : B) \\ \vdots\ \pi_1 \\ [H]^{\mathsf{a}_n}_{\mathsf{x}_n} \otimes (\overset{\pi}{\Rightarrow} a \Vdash^{\forall} B)\end{array}}{[H]^{\mathsf{a}_n}_{\mathsf{x}_n}}\ \mathsf{R}\Box$$

where $\pi_1$ is the derivation presented in the proof of Lemma 7.1 (b). Observe that $[H]^{\mathsf{a}_n}_{\mathsf{x}_n} \otimes (x_{n+1} \in a, \{x_{n+1} \in a_j\}_j, \{x_{n+1} : A_j\}_j \Rightarrow x_{n+1} : B)$ is, in fact, $[H_1]^{\mathsf{a}_n}_{\mathsf{x}_{n+1}} \otimes (x_{n+1} \in a, \{x_{n+1} \in a_j\}_j \Rightarrow\ )$.

Suppose that $a \Vdash^{\forall} B$ is principal at some point in $\pi$. Since $\mathsf{R} \Vdash^{\forall}$ is invertible, it can be eagerly applied and $\pi$ can be re-written as

$$\frac{\dfrac{[H]^{\mathsf{a}_n}_{\mathsf{x}_n} \otimes (\overset{\pi'}{x_{n+1} \in a \Rightarrow x_{n+1} : B})}{[H]^{\mathsf{a}_n}_{\mathsf{x}_n} \otimes (\Rightarrow a \Vdash^{\forall} B)}\ \mathsf{R} \Vdash^{\forall}}{[H]^{\mathsf{a}_n}_{\mathsf{x}_{n+1}}}\ \mathsf{R}\Box$$

where the application of the rule $\mathsf{R} \Vdash^{\forall}$ over $a \Vdash^{\forall} B$ is permuted down (and thus it does not appear in $\pi'$). This same argument can be applied to $\mathsf{dec}$ and $\mathsf{L} \Vdash^{\forall}$

over $x_{n+1} \in a$ and $x_{n+1} \in a_j$, respectively, obtaining the proof

$$\dfrac{\begin{array}{c}\pi'' \\ [H_1]^{\mathsf{a}_n}_{\mathsf{x}_{n+1}} \otimes (x_{n+1} \in a, \{x_{n+1} \in a_j\}_j \Rightarrow ) \\ \vdots\, \pi_1 \\ [H]^{\mathsf{a}_n}_{\mathsf{x}_n} \otimes ( \Rightarrow a \Vdash^{\forall} B)\end{array}}{[H]^{\mathsf{a}_n}_{\mathsf{x}_{n+1}}}\ \mathsf{R}\Box$$

According to Lemma 7.1 (c), $[H_1]^{\mathsf{a}_n}_{\mathsf{x}_{n+1}} \otimes (x_{n+1} \in a, \{x_{n+1} \in a_j\}_j \Rightarrow )$ is provable iff $[H_1]^{\mathsf{a}_n}_{\mathsf{x}_{n+1}}$ is provable. By inductive hypothesis, $H_1$ is provable in $\mathbf{H.E}^*$ with proof $\delta$. Hence $H$ is provable with proof

$$\dfrac{\begin{array}{c}\delta \\ G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Box B, \Delta \mid \Sigma \Rightarrow B\end{array}}{G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Box B, \Delta}\ \mathsf{R}\Box\mathsf{m}$$

$\Box$

We finish this section by illustrating the translation in the monotonic case.

**Example 7.1.** Consider the following derivation of the axiom $M$ in $\mathbf{H.M}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{H = \Box(A \wedge B), \langle A \wedge B\rangle \Rightarrow \Box A \mid A \wedge B, A, B \Rightarrow A}\ \mathsf{init}}{\Box(A \wedge B), \langle A \wedge B\rangle \Rightarrow \Box A \mid A \wedge B \Rightarrow A}\ \mathsf{L}\wedge}{\Box(A \wedge B), \langle A \wedge B\rangle \Rightarrow \Box A}\ \mathsf{R}\Box\mathsf{m}}{\Box(A \wedge B) \Rightarrow \Box A}\ \mathsf{L}\Box$$

This is mimicked in $\mathbf{LSE}^*$ by

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{S = a \rhd x_1, a \Vdash^{\forall} (A \wedge B), x_2 \in a, x_2 : A, x_2 : B \Rightarrow x_1 : \Box A, \overline{a} \Vdash^{\exists} (A \wedge B), x_2 : A}\ \mathsf{init}}{a \rhd x_1, a \Vdash^{\forall} (A \wedge B), x_2 \in a, x_2 : (A \wedge B) \Rightarrow x_1 : \Box A, \overline{a} \Vdash^{\exists} (A \wedge B), x_2 : A}\ \mathsf{L}\wedge}{a \rhd x_1, a \Vdash^{\forall} (A \wedge B), x_2 \in a \Rightarrow x_1 : \Box A, \overline{a} \Vdash^{\exists} (A \wedge B), x_2 : A}\ \mathsf{L}\Vdash^{\forall}}{a \rhd x_1, a \Vdash^{\forall} (A \wedge B) \Rightarrow x_1 : \Box A, \overline{a} \Vdash^{\exists} (A \wedge B), a \Vdash^{\forall} A}\ \mathsf{R}\Vdash^{\forall} \quad \pi}{a \rhd x_1, a \Vdash^{\forall} (A \wedge B) \Rightarrow x_1 : \Box A, \overline{a} \Vdash^{\exists} (A \wedge B)}\ \mathsf{R}\Box}{x_1 : \Box(A \wedge B) \Rightarrow x_1 : \Box A}\ \mathsf{L}\Box$$

where $\pi$ is

$$\dfrac{\dfrac{}{a \rhd x_1, x_2 \in \overline{a}, x_2 : A \Rightarrow x_1 : \Box A, \overline{a} \Vdash^{\exists} (A \wedge B)}\ \mathsf{M}}{a \rhd x_1, \overline{a} \Vdash^{\exists} A \Rightarrow x_1 : \Box A, \overline{a} \Vdash^{\exists} (A \wedge B)}\ \mathsf{L}\Vdash^{\exists}$$

Observe that $S = [U_H]^a_{x_1 x_2} \otimes (x_2 \in a \Rightarrow )$.

# 8 Discussion and Further Research

We have presented hypersequent calculi for the cube of classical non-normal modal logics extended with the axioms $T$, $P$, $D$, and the rules $RD^+_n$. Our main goal is to have calculi allowing direct countermodel construction and having optimal complexity, in particular for the logics lacking monotonicity.

Our calculi are based on the bi-neighbourhood semantics a variant of standard neighbourhood semantics, where each world has associated a set of pairs of neighbourhoods. Bi-neighbourhood semantics has been introduced in [9], they have however some analogy with the structure of bilattices introduced by Ginsberg [16] and extensively studied by Fitting (see e.g. [13]). It would be interesting to study the connections between these two kinds of structures. Here are some preliminary remarks. First of all we recall that a bilattice is a set equipped by two orderings $\leq_t$ and $\leq_k$ enforcing two lattice structures (with possibly some connecting conditions between the two). The basic construction for obtaining a bilattice is the "product" of two lattices $(C, \leq_1)$ and $(D, \leq_2)$: we can define the bilattice $\mathcal{B}(C, D) = (C \times D, \leq_t, \leq_k)$, by stipulating:

$(c_1, d_1) \leq_k (c_2, d_2)$ if $c_1 \leq_1 c_2$ and $d_1 \leq_2 d_2$ and
$(c_1, d_1) \leq_t (c_2, d_2)$ if $c_1 \leq_1 c_2$ and $d_2 \leq_2 d_1$.

The intuition behind the construction is similar to bi-neighbourhood one: "*Suppose we think of a pair* $(c, d) \in \mathcal{B}(C, D)$ *as representing two independent judgements concerning the truth of some statement: c represents our degree of belief for the statement; d represents our degree of belief against it...*" [13]. In the bi-neighbourhood semantics, we can interpret the order $\leq_k$ as component-wise inclusion on pairs, so that we get that if a pair $(\alpha, \beta)$ "witnesses" a formula $\Box A$ (that is $\alpha \subseteq [\![A]\!]_{\mathcal{M}} \subseteq \mathcal{W} \setminus \beta$) and $(\alpha', \beta') \leq_k (\alpha, \beta)$, then also $(\alpha', \beta')$ "witnesses" $\Box A$. Moreover, we notice that the condition (C) *coincides* with the meet operation (w.r.t. $\leq_t$) in bilattices as $(c_1, d_1) \sqcap_t (c_2, d_2) = (c_1 \sqcap c_2, d_1 \sqcup d_2)$, so that in the bi-neighbourhood semantics (C) asserts the closure of $\mathcal{N}(w)$ with respect to $\sqcap_t$. However, $\mathcal{N}(w)$ does not have in general a lattice structure, neither with respect to $\leq_k$, nor with respect to $\leq_t$. In further research we will study whether the relation between the two kinds of structures can go deeper.

As we have already observed, not many works in the literature present proof systems both allowing countermodel construction and enjoying optimal complexity. In this respect, the nested sequent calculus for **M** proposed in [26] achieves both goals it allows for both direct countermodel construction and can be adapted for optimal complexity, similarly to what we did in Section 5. However, as we explained, the nested-sequent structure is of no help for non-monotonic logics. Apart from the flat hypersequent structure considered here, the main difference to the calculi considered in this work is that here we include the structural connective for the modal box in the antecedent of the sequents, thus enabling relatively simple rules for distribution of boxes over conjunction. In contrast, in the nested sequent calculi of [26], the structural connective for box is in the consequent, in line with standard nested sequent calculi for normal modal logics. Additionally, since the logics there also contain normal modal logic **K**, they are of PSPACE complexity.

Furthermore, optimal decision procedures for all logics of the classical cube are presented in [18]. The procedures reduce validity/satisfiability in each modal logic to a set of SAT problems, to be handled by a SAT solver; despite their efficiency, the procedures provide neither "proofs", nor countermodels, whence having a different aim from the calculi of this work. Our hypersequent calculi have nonetheless an interest for automated reasoning: for systems within the classical cube, they have been implemented in the theorem prover HYPNO [10].

Two issues remain open: the first one is how to extend the present framework to deal with the axioms 4, 5, $B$ of normal modal logic, in particular in the

non-monotonic case. This issue has both a semantical and a proof-theoretical side. Since our calculi are based on the bi-neighbourhood semantics, this investigation presupposes an extension of bi-neighbourhood semantics itself to cover these axioms, which is not entirely straightforward (see [7]). From the proof-theoretical side, the simple hypersequent structure with blocks used in this work might no longer be sufficient for capturing these stronger systems; since the axioms 4, 5, $B$ concern nested modalities, a more complex structure comprising a (mutual) *nesting* of sequent and blocks might be needed.

Another issue is the one of interpolation: [33] presents a general result on uniform interpolation for rank-1 logics, which would cover all examples considered here. However, since there seem to be some issues with this result [37], and since the construction of the interpolants is not fully explicit there, it is worth continuing the exploration of proof theoretic ways of showing interpolation results. In [30] a constructive proof of Craig interpolation is provided for a good part of the logics considered in this work, but not for non-monotonic logics with $C$. Could our calculi be used to cover these missing cases, perhaps using methods like those of [24]? We intend to investigate this issue in future work.

# 9 Conclusions

We have presented hypersequent calculi for the cube of classical non-normal modal logics extended with axioms $T$, $P$, $D$, and rules $RD_n^+$. Apart from the distinction between monotonic and non-monotonic systems, the calculi are modular. They also have a natural, and "almost internal" interpretation, as each component of a hypersequent can be read as a formula of the language. We have shown that the hypersequent calculi have good structural properties, in particular they enjoy cut elimination. The calculi provide a decision procedure, which is of optimal (coNP) complexity for logics without $C$. Moreover, from a failed proof we can easily extract a countermodel (of polynomial size for logics without axiom $C$) in the bi-neighbourhood semantics, whence by an easy transformation also in the standard one. Finally, the hypersequent calculi can be embedded in the labelled calculi of [9] for the classical cube, providing thereby a kind of "compact encoding" of derivations in the latter.

All in all, the structure of our calculi, namely hypersequents with blocks, provides an adequate framework for extracting countermodels from a single failed proof, ensuring, at the same time, good computational and structural properties, as well as modularity. In particular, we believe that this structure is likely the simplest and the most adequate having these features in the non-monotonic case. Our future main goal is to extend extend our calculi to other non-normal modal logics, counterparts of the normal modal logic cube and to obtain constructive proofs of interpolation for all non-normal modal logics.

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