Labelled Connection-based Proof Search for Multiplicative Intuitionistic Linear Logic

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Abstract

We propose a connection-based characterization for multiplicative intuitionistic linear logic (MILL) which is based on labels and constraints that capture Urquhart's possible world semantics of the logic. We first briefly recall the purely syntactic sequent calculus for MILL, which we call LMILL. Then, in the spirit of our previous results on the Logic of Bunched Implications (BI), we present a connection-based characterization of MILL provability. We show its soundness and completeness without the need for any notion of multiplicity. From the characterization, we finally propose a labelled sequent calculus for MILL.

1 Introduction

In previous works we have developed connection-based characterizations of validity in non-classical logics like the Logic of Bunched Implications (BI) [2] and Bi-intuitionistic logic [3]. They are based on specific concepts like *labels* and *constraints* in order to capture the model semantics and then the semantic interactions between connectives in such logics. It is an alternative approach to the standard view of connection calculi for non-classical logics that are based on the notion of *prefixes*. This notion allows one to capture the non-permutabilities of the sequent calculi rules and has been developed and improved in the context of intuitionistic logic [9, 10] but also of modal logics [8]. There exist connection-based characterizations and related connection methods for multiplicative (commutative) linear logic (MLL) [1, 6] but, as far as we know, not for multiplicative intuitionistic linear logic (MILL). The connection-based characterization proposed for fragments of Linear Logic [4] like MLL or MELL is based on particular prefixes and substitutions dedicated to these logics [6, 5]. In order to extend or adapt it to MILL, it would be necessary to define and consider what could be called intuitionistic and linear prefixes, which could be difficult to deal with.

Our approach consists in specializing our above-mentioned results for BI to the Multiplicative Intuitionistic Linear Logic (MILL) and then define and illustrate a connection-based characterization of provability for MILL that deals with specific labels and constraints. Then we generate semantic structures from MILL's Urquhart's semantics [12] and develop a characterization of provability from labels and constraints that capture this semantics. It could be seen as a generalization of the prefixes more appropriate to connection-based proof search in resource logics like BI logic or Linear Logic. Since BI is conservative over MILL [7], a connection-based characterization for MILL can be obtained, by restriction of the previous characterization for BI, to the multiplicative connectives. In this case, we have to take into account the notion of multiplicity and also global conditions on the paths. The characterization for MILL proposed here does not deal with multiplicity and only considers local conditions on the paths. In addition we define a labelled sequent calculus for MILL, called GMILL and prove its soundness and also its completeness by translation of MILL proofs to GMILL proofs.

$$\frac{}{\mathrm{A} \vdash \mathrm{A}}\,ax \qquad \frac{\Gamma \vdash \mathrm{C}}{\Gamma, 1 \vdash \mathrm{C}}\,1_L \qquad \frac{}{\vdash 1}\,1_R$$

$$\frac{\Gamma, \mathrm{A}, \mathrm{B}, \Delta \vdash \mathrm{C}}{\Gamma, \mathrm{A} * \mathrm{B}, \Delta \vdash \mathrm{C}}\,*_L \qquad \frac{\Gamma \vdash \mathrm{A} \quad \Delta \vdash \mathrm{B}}{\Gamma, \Delta \vdash \mathrm{A} * \mathrm{B}}\,*_R$$

$$\frac{\Gamma \vdash \mathrm{A} \quad \Delta, \mathrm{B} \vdash \mathrm{C}}{\Gamma, \mathrm{A} \mathrel{-\!\!*} \mathrm{B}, \Delta \vdash \mathrm{C}}\,\mathrel{-\!\!*}_L \qquad \frac{\Gamma, \mathrm{A} \vdash \mathrm{B}}{\Gamma \vdash \mathrm{A} \mathrel{-\!\!*} \mathrm{B}}\,\mathrel{-\!\!*}_R$$

Figure 1: Sequent Calculus for MILL

2 Multiplicative Intuitionistic Linear Logic

The propositional language of MILL consists of a denumerable set $L = \mathrm{P}, \mathrm{Q}, \ldots$ of propositional letters, the multiplicative unit 1 and the multiplicative connectives $*$ and $\mathrel{-\!\!*}$. $\mathcal{P}(L)$, the collection of MILL propositions over L, is given by the following inductive definition:

$$\mathrm{A} ::= \mathrm{P} \mid 1 \mid \mathrm{A} * \mathrm{A} \mid \mathrm{A} \mathrel{-\!\!*} \mathrm{A}.$$

Let us remark that since the forthcoming connection-based characterization of MILL-provability is inspired by our previous work on BI [2], we do not use the more widespread symbols $\otimes$ and $\multimap$ to denote multiplicative conjunction and implication and rather stick with the star and magic-wand notations of these connectives.

Judgements of MILL are sequents of the form $\Gamma \vdash \mathrm{A}$, where A is a proposition and Γ, called the context, is a (possibly empty) multiset of formulas.

The standard sequent calculus for MILL, which we call LMILL[1], is given in Figure 1. One difficulty with such a calculus lies in the fact that the rules for left-implication and right-conjunction both require context splitting from conclusion to premises. Making relevant choices when context-splitting is required is crucial for the efficiency of backward proof-search.

The semantics we use for MILL models is a possible worlds semantics à la Kripke, mainly inspired from the operational semantics of Urquhart [12]. Let us recall it briefly.

Definition 1 (MILL-frame)**.** *A* MILL-frame *is a partially ordered commutative monoid* $\mathcal{M} = \langle M, \cdot, e, \sqsubseteq \rangle$*, in which* M *is a set of worlds and* $\sqsubseteq$ *is compatible with* $\cdot$*,* i.e.*:*

$$\forall m \forall n \in M.\ \textit{if } m \sqsubseteq n \textit{ and } m' \sqsubseteq n' \textit{ then } m \cdot m' \sqsubseteq n \cdot n'.$$

Definition 2 (MILL-interpretation)**.** *A* MILL-intepretation *is a function* $[\![-]\!] : L \to \mathcal{P}(M)$ *that satisfies Kripke monotonicity,* i.e.*:*

$$\forall m, n \in M.\ \textit{if } m \sqsubseteq n \textit{ and } m \in [\![\mathrm{P}]\!] \textit{ then } n \in [\![\mathrm{P}]\!].$$

Definition 3 (MILL-model)**.** *Let* $\mathcal{P}(L)$ *be the collection of MILL propositions over a language* L *of propositional letters, a* MILL-model *is a structure* $\mathcal{R} = \langle M, \cdot, e, \sqsubseteq, [\![-]\!], \models \rangle$*, in which* $\langle M, \cdot, e, \sqsubseteq \rangle$ *is a MILL-frame,* $[\![-]\!]$ *is a MILL-interpretation, and* $\models$ *is a forcing relation on* $M \times \mathcal{P}(L)$ *satisfying the following conditions:*

- $m \models \mathrm{P}$ *iff* $m \in [\![\mathrm{P}]\!]$

[1] In the spirit of LJ and LK for intuitionistic and classic logic, although LMILL has no labels.

- $m \models 1$ *iff* $e \sqsubseteq m$
- $m \models \mathrm{A} * \mathrm{B}$ *iff there exist* $n_1, n_2 \in M$ *such that* $n_1 \cdot n_2 \sqsubseteq m$, $n_1 \models \mathrm{A}$ *and* $n_2 \models \mathrm{B}$
- $m \models \mathrm{A} \multimap \mathrm{B}$ *iff, for all* $n_1, n_2 \in M$, *if* $n_1 \models \mathrm{A}$ *and* $m \cdot n_1 \sqsubseteq n_2$ *then* $n_2 \models \mathrm{B}$.

Definition 4 (MILL-validity)**.** *Let* $\mathcal{R}$ *be MILL-model. A formula* A *is* valid in $\mathcal{R}$, *written* $\mathcal{R} \models \mathrm{A}$, *iff* $e \models \mathrm{A}$. A *is* valid, *written,* $\models \mathrm{A}$, *iff* $\mathcal{R} \models \mathrm{A}$ *for all models* $\mathcal{R}$. *A finite set of formulas* $\{\mathrm{A}_1, \ldots, \mathrm{A}_n\}$ entails *a formula* B, *written* $\mathrm{A}_1, \ldots, \mathrm{A}_n \models \mathrm{B}$, *iff* $\models (\mathrm{A}_1 * \ldots * \mathrm{A}_n) \multimap \mathrm{B}$.

Theorem 1 (Soundness and Completeness)**.** *For all formulas* A, $\models \mathrm{A}$ *iff* $\vdash \mathrm{A}$.

3 Labelled Connections for MILL

The connection-based characterization of MILL-provability we define in this paper is based on *labels* and *constraints* that capture the semantic properties of MILL-frames instead of capturing the syntactic properties (such as permutabilities, context-splitting or linearity) of the purely syntactic sequent calculus LMILL as the standard prefix-based approach does for IL or MLL [5, 6, 13]. We already successfully used a similar approach for BI [2], and since BI is conservative w.r.t. MILL [7], the characterization in [2] also applies to MILL. However, although the characterization for MILL presented in this paper can indeed be seen as a refinement of the one given for BI, its improvements are two-fold: firstly, it does not need any notion of multiplicity and, secondly, it is local in that the conditions for characterizing provability are not stated w.r.t. the global set of atomic paths but w.r.t. each atomic path individually. The locality of the characterization is a key step towards implementing a memory-space efficient depth-first path-reduction strategy in a future connection-based prover for MILL. Indeed global conditions would require us to keep the whole set of atomic paths in memory while checking provability conditions. Since the number of atomic paths can grow exponentially large with the size of a formula, local conditions are preferable if one wants to make a more efficient use of the memory space (linear in the size of the initial formula).

3.1 Labels and Constraints

Given an alphabet C (for example $a, b, c, \cdots$), C_0, the set of *atomic labels over* C, is defined as the set C extended with the unit symbol ϵ. We then define $\mathcal{L}C$, the set of *labels over* C, as the smallest set containing C_0, and closed under composition ($x, y \in \mathcal{L}C$ implies $xy \in \mathcal{L}C$). Labels are considered up to associativity, commutativity and identity w.r.t. ϵ. Therefore, *aabcc*, *cbaca* and *cbcaaε* are simply regarded as equivalent.

A *label constraint* is an expression $x \leq y$ where x and y are labels. A constraint of the form $x \leq x$ is called an *axiom* and we write $x = y$ to express that $x \leq y$ and $y \leq x$. We use the following inference rules for reasoning on constraints:

$$\frac{}{x \leq x}\,R \qquad \frac{x \leq z \quad z \leq y}{x \leq y}\,T \qquad \frac{x \leq y \quad x' \leq y'}{xx' \leq yy'}\,F$$

The R and T rules formalize the reflexivity and transitivity of $\leq$ while the F rule corresponds to the functoriality (also called compatibility) of label-composition w.r.t. $\leq$. In this formal system, given a constraint k and a set of constraints H, we write $H \mid\!\approx k$ if there is a deduction of k from H. The notation $H \mid\!\approx K$, where K is a non-empty set of constraints, means that for all $k \in K$, $H \mid\!\approx k$.

3.2 Labelled Indexed Formula Tree

Here we recall the standard notions coming from previous matrix-characterizations of provability [6, 13]. A *decomposition tree* of a formula A is its representation as a syntactic tree with nodes called *positions*. A position u exactly identifies a subformula of A denoted $f(u)$. An *atomic position* is a position for an atomic formula. If u is a non-atomic position the principal connective of $f(u)$ is denoted $c(u)$. Moreover such a position corresponds to an internal node and we denote $[u]_i$ with $i \in \{1, 2\}$ the position of the i-th child of the node corresponding to u and then $[u]_\star = \{ v \mid (\exists i \in \mathbb{N})(v = [u]_i) \}$. If u is not a root position we say that u is of rank $r(u) = i$ if u is the i-th child of its father position denoted by $[u]_0$.

The decomposition tree induces a partial order $\ll$ on the positions such that the root is the least element and if $u \ll v$ then u dominates v in the tree or in the formula (from now on we do not distinguish a formula A from its decomposition tree). Then we denote $[u]\uparrow$ the set $\{ v \mid v \in \mathrm{A} \textit{ and } v \ll u \}$ of upward positions of u and $[u]\downarrow$ the set $\{ v \mid v \in \mathrm{A} \textit{ and } u \ll v \}$ of its downward positions. The notations $[\cdot]\uparrow$ and $[\cdot]\downarrow$ are easily generalized to a set s of positions by $[s]\uparrow = \{ u \mid u \in [v]\uparrow \textit{ and } v \in s \}$ and $[s]\downarrow = \{ u \mid u \in [v]\downarrow \textit{ and } v \in s \}$.

For each position, we assign a polarity $pol(u)$ but also a principal type $ptyp(u)$ and a secondary type $styp(u)$. Therefore, we have different principal types depending on the connective and the associated polarity. We define two principal types named π_α, π_β. Given a set of positions p, $P_\alpha(p) = \{ u \mid u \in p \text{ and } ptyp(u) = \pi_\alpha \}$ is the set of positions of type π_α and $P_\beta(p) = \{ u \mid u \in p \text{ and } ptyp(u) = \pi_\beta \}$ is the set of positions of type π_β.
Moreover we consider the following sets of secondary type positions: for i in $\{1, 2\}$, $S_{\alpha i}(p) = \{ u \mid u \in p \text{ and } styp(u) = \pi_{\alpha i} \}$, $S_{\beta_i}(p) = \{ u \mid u \in p \text{ and } styp(u) = \pi_{\beta_i} \}$ and $S_\alpha(p) = S_{\alpha 1}(p) \cup S_{\alpha 2}(p)$, $S_\beta(p) = S_{\beta_1}(p) \cup S_{\beta_2}(p)$.

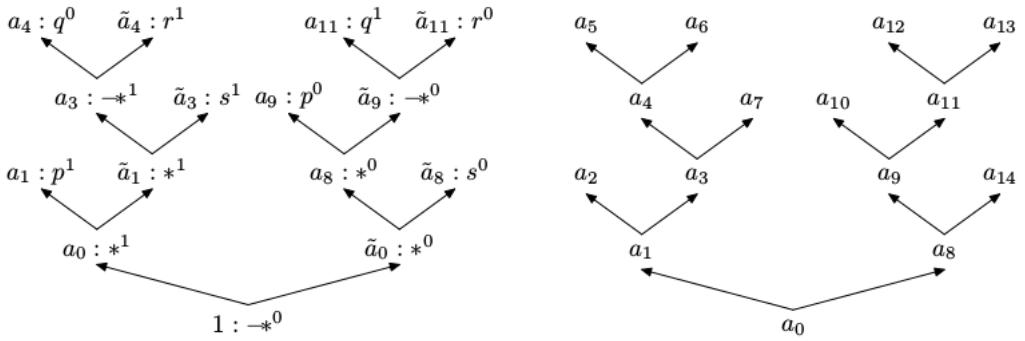
Depending on the principal type, we associate a label $slab(u)$ and sometimes a constraint $kon(u)$ to a position u. Such a label is either a position or a position with a tilde in order to identify the formula that introduced the label. We define constraints in order to capture context-splitting. The *labelled signed formula* $lsf(u)$ of a position u is a triple $(slab(u), f(u), pol(u))$ and is denoted $f(u)^{pol(u)} : slab(u)$. The construction of an indexed formula tree is obtained by inductively applying the rules described in Figure 2.

$lsf(u)$	$ptyp(u)$	$kon(u)$	$lsf(u_1)$	$lsf(u_2)$
$(\mathrm{A} \multimap\!\!* \mathrm{B})^0 : x$	π_α	$xu \leq \tilde{u}$	$\mathrm{A}^1 : u$	$\mathrm{B}^0 : \tilde{u}$
$(\mathrm{A} * \mathrm{B})^1 : x$	π_α	$u\tilde{u} \leq x$	$\mathrm{A}^1 : u$	$\mathrm{B}^1 : \tilde{u}$
$(\mathrm{A} \multimap\!\!* \mathrm{B})^1 : x$	π_β	$xu \leq \tilde{u}$	$\mathrm{A}^0 : u$	$\mathrm{B}^1 : \tilde{u}$
$(\mathrm{A} * \mathrm{B})^0 : x$	π_β	$u\tilde{u} \leq x$	$\mathrm{A}^0 : u$	$\mathrm{B}^0 : \tilde{u}$

Figure 2: Signed formulae for MILL

For a given formula A the root position a_0 has a polarity $pol(a_0) = 0$, a label $slab(a_0) = \epsilon$ and the signed formula $(\mathrm{A})^0 : \epsilon$ where ϵ is the identity of label composition. u_1 and u_2 are respectively the first and second subpositions. The principal type of a position u depends on its principal connective and polarity.

The constraints associated to π_α-positions are called *assertions* while those associated to π_β-positions are called *requirements*. The atomic labels introduced by positions of principal type π_α (resp. π_β) are called constants (resp. variables). Given a set of positions p, the associated sets of assertions and requirements are $\mathcal{K}_\alpha(p) = \{ kon(u) \mid u \in P_\alpha(p) \}$ and $\mathcal{K}_\beta(p) = \{ kon(u) \mid u \in P_\beta(p) \}$ respectively. The associated sets of constants and variables are then



u	$pol(u)$	$f(u)$	$ptyp(u)$	$styp(u)$	$slab(u)$	$kon(u)$
a_0	0	$(\mathrm{P} * ((\mathrm{Q} -\!\!* \mathrm{R}) * \mathrm{S})) -\!\!* ((\mathrm{P} * (\mathrm{Q} -\!\!* \mathrm{R})) * \mathrm{S})$	π_α	$-$	ϵ	$\epsilon a_0 \leq \tilde{a}_0$
a_1	1	$\mathrm{P} * ((\mathrm{Q} -\!\!* \mathrm{R}) * \mathrm{S})$	π_α	$\pi_{\alpha 1}$	a_0	$a_1\tilde{a}_1 \leq a_0$
a_2	1	P	$-$	$\pi_{\alpha 1}$	a_1	$-$
a_3	1	$(\mathrm{Q} -\!\!* \mathrm{R}) * \mathrm{S}$	π_α	$\pi_{\alpha 2}$	$\tilde{a}_1$	$a_3\tilde{a}_3 \leq \tilde{a}_1$
a_4	1	$\mathrm{Q} -\!\!* \mathrm{R}$	π_β	$\pi_{\alpha 1}$	a_3	$a_3a_4 \leq \tilde{a}_4$
a_5	0	Q	$-$	π_{β_1}	a_4	$-$
a_6	1	R	$-$	π_{β_2}	$\tilde{a}_4$	$-$
a_7	1	S	$-$	$\pi_{\alpha 2}$	$\tilde{a}_3$	$-$
a_8	0	$(\mathrm{P} * (\mathrm{Q} -\!\!* \mathrm{R})) * \mathrm{S}$	π_β	$\pi_{\alpha 2}$	$\tilde{a}_0$	$a_8\tilde{a}_8 \leq \tilde{a}_0$
a_9	0	$\mathrm{P} * (\mathrm{Q} -\!\!* \mathrm{R})$	π_β	π_{β_1}	a_8	$a_9\tilde{a}_9 \leq a_8$
a_{10}	0	P	$-$	π_{β_1}	a_9	$-$
a_{11}	0	$\mathrm{Q} -\!\!* \mathrm{R}$	π_α	π_{β_2}	$\tilde{a}_9$	$\tilde{a}_9a_{11} \leq \tilde{a}_{11}$
a_{12}	1	Q	$-$	$\pi_{\alpha 1}$	a_{11}	$-$
a_{13}	0	R	$-$	$\pi_{\alpha 2}$	$\tilde{a}_{11}$	$-$
a_{14}	0	S	$-$	π_{β_2}	$\tilde{a}_8$	$-$

Figure 3: Indexed formula tree of $(\mathrm{P} * ((\mathrm{Q} -\!\!* \mathrm{R}) * \mathrm{S})) -\!\!* ((\mathrm{P} * (\mathrm{Q} -\!\!* \mathrm{R})) * \mathrm{S})$

defined as $\Sigma_\alpha(p) = \{ x \mid x \text{ occurs in } \mathcal{K}_\alpha(p) \}$ and $\Sigma_\beta(p) = \{ x \mid x \text{ occurs in } \mathcal{K}_\beta(p) \}$. We set $\Sigma_{\alpha\beta}(p) = \Sigma_\alpha(p) \cup \Sigma_\beta(p)$ and define $\mathcal{L}_\alpha(p)$, $\mathcal{L}_\beta(p)$ and $\mathcal{L}_{\alpha\beta}(p)$ as the sets of atomic and compound labels generated by $\Sigma_\alpha(p)$, $\Sigma_\beta(p)$ and $\Sigma_{\alpha\beta}(p)$ respectively. For readability, we omit the set of positions p in notations whenever p is the set $\mathcal{P}os$ of all positions.

3.2.1 An Example (part 1)

Let us consider the formula $\mathrm{A} \equiv (\mathrm{P} * ((\mathrm{Q} -\!\!* \mathrm{R}) * \mathrm{S})) -\!\!* ((\mathrm{P} * (\mathrm{Q} -\!\!* \mathrm{R})) * \mathrm{S})$ that is represented as a syntax tree each node of which being identified with a position (see the tree at the righthand side of Figure 3). Moreover, we can associate an indexed formula tree (with labelled signed formulae as nodes), inductively built from $(\mathrm{A})^0 : \epsilon$ and the rules of Figure 2. This tree is the one at the lefthand side of Figure 3. In parallel, we have the generation of constraints for the positions of principal type π_α and π_β (see $kon(u)$ in the table of Figure 3). Then, in this case we can deduce that $P_\alpha = \{ a_0, a_1, a_3, a_{11} \}$, $P_\beta = \{ a_4, a_8, a_9 \}$, $S_{\alpha 1} = \{ a_1, a_2, a_4, a_{12} \}$, $S_{\beta_1} = \{ a_5, a_9, a_{10} \}$, $S_{\alpha 2} = \{ a_3, a_7, a_8, a_{13} \}$, $S_{\beta_2} = \{ a_6, a_{11}, a_{14} \}$. In addition the assertions and requirements are

$$\mathcal{K}_\alpha = \{ \epsilon a_0 \leq \tilde{a}_0, a_1\tilde{a}_1 \leq a_0, a_3\tilde{a}_3 \leq \tilde{a}_1, \tilde{a}_9 a_{11} \leq \tilde{a}_{11} \}$$
$$\mathcal{K}_\beta = \{ a_8\tilde{a}_8 \leq \tilde{a}_0, a_9\tilde{a}_9 \leq a_8, a_3a_4 \leq \tilde{a}_4 \}$$

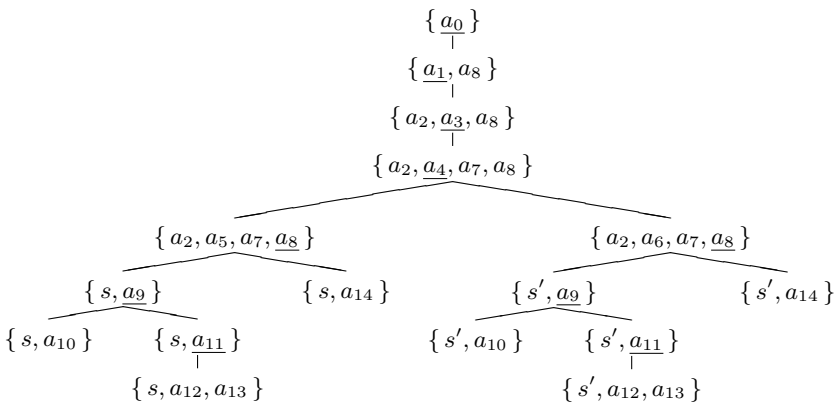


Figure 4: Path reduction with $s = a_2, a_5, a_7$ and $s' = a_2, a_6, a_7$

The constants and variables are

$$\Sigma_\alpha = \{ a_0, \tilde{a}_0, a_1, \tilde{a}_1, a_3, \tilde{a}_3, a_{11}, \tilde{a}_{11} \} \qquad \Sigma_\beta = \{ a_4, \tilde{a}_4, a_8, \tilde{a}_8, a_9, \tilde{a}_9 \}.$$

3.3 Paths and Connections

In this section, we adapt the standard notions of path, connection and spanning set of connections in the context of labels and constraints.

Definition 5 (Paths)**.** *Let* A *be an indexed formula. The set of* paths *in* A *is inductively defined as follows:*

1. $\{ a_0 \}$ *is a path, where* a_0 *is the root position.*
2. *If* s *is a path such that* $u \in s$ *then*
   - *if* $ptyp(u) \in \{ \pi_\alpha \}$ *then* $s \backslash \{ u \} \cup \{ [u]_1, [u]_2 \}$ *is a path,*
   - *if* $ptyp(u) \in \{ \pi_\beta \}$ *then* $s \backslash \{ u \} \cup \{ [u]_1 \}$ *and* $s \backslash \{ u \} \cup \{ [u]_2 \}$ *are paths*[2].

We say that a path s' in A is obtained from a path s by *reduction* on the indexed position u if it results from s using the second clause of Definition 5. An *atomic path* is a path that only contains atomic indexed positions. Consequently an atomic path is non-reducible and is always a leaf of a path reduction tree. A *configuration* of A is a finite set of paths in A.

Definition 6 (Reduction)**.** *A* reduction *of an indexed formula* A *is a finite sequence* $(\mathcal{S}_i)_{1 \leq i \leq n}$ *of configurations in* A *such that* $\mathcal{S}_{i+1}$ *is obtained from* $\mathcal{S}_i$ *by reduction of a position* u *in a path* s *of* $\mathcal{S}_i$ *following Definition 5. We say that* $\mathcal{S}_{i+1}$ *is obtained by reduction of* $\mathcal{S}_i$ *of* u *in* s*. A reduction* $(\mathcal{S}_i)_{1 \leq i \leq n}$ *is said* atomic *if all the paths of* $\mathcal{S}_n$ *are atomic.*

Definition 7 (Connection)**.** *Let* A *be an indexed formula, a* connection c *in* A *is:*

1. *a pair* $\langle u, v \rangle$ *of atomic positions such that* $f(u) = f(v)$, $pol(u) = 1$ *and* $pol(v) = 0$, *or*
2. *a pair* $\langle a_0, v \rangle$ *such that* $f(v) = 1$ *and* $pol(v) = 0$.

[2] Let us remark that the branching indicates non-determinism.

The first case corresponds to an atomic axiom rule (ax) *in LMILL, while the second one corresponds to the rule* 1_R*. Let us also note that the first position of a connection is the one with the positive polarity.*

We denote $\mathcal{C}on$ the set of connections in A. The constraint $kon(c)$ associated to a connection $c = \langle u, v \rangle$ is defined as $kon(c) = slab(u) \leq slab(v)$. For a 1-connection $c = \langle a_0, v \rangle$ we have $kon(\langle a_0, v \rangle) = \epsilon \leq slab(v)$ since $slab(a_0) = \epsilon$. In order to distinguish these constraints from the assertions and requirements they are called *connection constraints*. Moreover, the notions of upward and downward positions are extended to connections as follows: $c\uparrow = \{ u, v \}\uparrow$ and $c\downarrow = \{ u, v \}\downarrow$.

Definition 8 (MILL-cover)**.** *Let* A *be an indexed formula, a connection* $c = \langle u, v \rangle$ *in* A covers *a path* s *in* A*, denoted* $c \succ s$*, if* $v \in s$ *and* $(u \neq a_0 \Rightarrow u \in s)$*. Let* S *be a set of paths in* A*, a* cover *of* S *is a set* $C = \{ (s, \langle u, v \rangle) \mid s \in S$ *and* $\langle u, v \rangle \in \mathcal{C}on$ *and* $\langle u, v \rangle \succ s \}$ *such that*

$$(s, \langle u, v \rangle) \in C \text{ and } (s, \langle u', v' \rangle) \in C \Rightarrow u = u' \text{ and } v = v'.$$

A cover of A *is a cover of the set of atomic paths in* A*.*

3.3.1 An Example (part 2)

The reduction of the initial path $\{ a_0 \}$ results in six atomic paths as depicted in Figure 4. At each step, we indicate the position which is reduced with an underscore. For conciseness, we write s and s' as shortcuts for a_2, a_5, a_7 and a_2, a_6, a_7. The set $C =$

$$\{ (s_1, \langle a_2, a_{10} \rangle), (s_2, \langle a_{12}, a_5 \rangle), (s_3, \langle a_7, a_{14} \rangle), (s_4, \langle a_2, a_{10} \rangle), (s_5, \langle a_6, a_{13} \rangle), (s_6, \langle a_7, a_{14} \rangle) \}$$

covers all atomics paths. Indeed, $\langle a_2, a_{10} \rangle$ covers paths s_1 and s_4, $\langle a_{12}, a_5 \rangle$ covers the path s_2, $\langle a_7, a_{14} \rangle$ covers the path s_3 and s_6, $\langle a_6, a_{13} \rangle$ covers the path s_5. We observe that connections $\langle a_2, a_{10} \rangle$ and $\langle a_7, a_{14} \rangle$ cover two atomic paths at the same time.

3.4 Characterizing MILL-Provability

In this section we define a connection-based characterization of MILL-provability which relies on the notions of substitution and certification.

Definition 9 (Substitution)**.** *Let* A *be an indexed formula. A* substitution for A *is an application* $\sigma : \Sigma_\beta \to \mathcal{L}_\alpha$*, that can be extended to labels and constraints as follows:*

- $x\sigma = x$ *if* x *is a constant or if* $x = \epsilon$*,*
- $(x \circ y)\sigma = x\sigma \circ y\sigma$*,*
- $(x \leq y)\sigma = x\sigma \leq y\sigma$*.*

Moreover a substitution σ for an indexed formula A induces an *instantiation relation* on indexed positions, denoted $\prec$, such that

$$(\forall u, v \in \mathcal{P}os)(u \prec v \text{ iff } v \in P_\beta \text{ and } (u \subseteq v\sigma \text{ or } \tilde{u} \subseteq v\sigma).$$

Definition 10 (Certification)**.** *Let* A *be an indexed formula. A* certification for A *is an application* $\gamma : P_\beta \to \wp(P_\alpha)$ *that associates a set of* π_α*-positions with any* π_β*-position in* A*.*

A certification γ for an indexed formula A induces a *deduction relation* on indexed positions, denoted $\sqsubset$, such that

$$(\forall u, v \in \mathcal{P}os)(u \sqsubset v \text{ iff } v \in P_\beta \text{ and } u \in v\gamma).$$

An expression $u \sqsubset v$ means that v is deduced from u (in A). The relations of domination, instantiation and deduction induce a *reduction relation* $\lhd = (\ll \cup \prec \cup \sqsubset)^+$ where $(\cdot)^+$ represents the transitive closure. An expression $u \lhd v$ means that u must be reduced before v (in A). Now we can express the provability conditions in terms of connections.

Definition 11 (Complementarity)**.** *Let s be a path in an indexed formula* A *and σ be a substitution, a connection c in* A *is* complementary in s under σ *if $c \succ s$ and $\mathcal{K}_\alpha(s\uparrow)\sigma \models\!\approx kon(c)\sigma$. A path s is* complementary under σ *if there exists a connection that is complementary in s under σ. A cover C of a set of paths in* A *is* complementary under σ *if* $(\forall(s,c) \in C)$ *c is complementary in s under σ.*

Definition 12 (Provability)**.** *A formula* A *is* provable *if there exist a cover C of the set of atomic paths of* A*, a substitution σ and a certification γ for* A *such that:*

1. *the reduction relation $\lhd$ is irreflexive,* $(C1)$
2. $\forall(s, \langle u, v \rangle) \in C, \forall w \in P_\beta(s\uparrow), w\gamma \subseteq P_\alpha(s\uparrow),$ $(C2)$
3. $\forall(s, \langle u, v \rangle) \in C, \forall w \in P_\beta(s\uparrow), k(w\gamma)\sigma \models\!\approx k(w)\sigma,$ $(C3)$
4. $\forall(s, \langle u, v \rangle) \in C, \forall x \in \Sigma_\beta(s\uparrow), x\sigma \in \mathcal{L}_\alpha(s\uparrow),$ $(C4)$
5. $\forall(s, \langle u, v \rangle) \in C, \langle u, v \rangle$ *is complementary in s under σ.* $(C5)$

The first condition induces the acyclicity of the graph associated to $\lhd$ and then the existence of a reduction (decomposition) order of the formula A that respects the precedence constraints between π_α and π_β positions. The second and third conditions ensure that, in an atomic path s, every requirement introduced by a position of principal type π_β must be introduced before the two positions of the connection that makes the path s complementary and should be certified by assertions corresponding to positions of principal type π_α that can be reduced before this requirement in a reduction from the initial path $\{ a_0 \}$ to s. In a similar way the fourth condition means that each variable, introduced before reaching the connection that makes an atomic path complementary, is instantiated by a label composed from constants that can be reduced before this variable in a reduction from the initial path $\{ a_0 \}$ to s.

3.4.1 An Example (part 3)

The reduction path process from $\{ a_0 \}$ provides the following atomic paths:
$s_1 = \{ a_2, a_5, a_7, a_{10} \}$, $s_2 = \{ a_2, a_5, a_7, a_{12}, a_{13} \}$, $s_3 = \{ a_2, a_5, a_7, a_{14} \}$, $s_4 = \{ a_2, a_6, a_7, a_{10} \}$, $s_5 = \{ a_2, a_6, a_7, a_{12}, a_{13} \}$ and $s_6 = \{ a_2, a_6, a_7, a_{14} \}$.
From the following cover $C =$

$$\{ (s_1, \langle a_2, a_{10} \rangle), (s_2, \langle a_{12}, a_5 \rangle), (s_3, \langle a_7, a_{14} \rangle), (s_4, \langle a_2, a_{10} \rangle), (s_5, \langle a_6, a_{13} \rangle), (s_6, \langle a_7, a_{14} \rangle) \}$$

we generate the set of constraints:

$$\mathcal{K}_C = \{ (s_1, a_1 \leq a_9), (s_2, a_{11} \leq a_4), (s_3, \tilde{a}_3 \leq \tilde{a}_8), (s_4, \tilde{a}_1 \leq \tilde{a}_9), (s_5, \tilde{a}_4 \leq \tilde{a}_{11})(s_6, \tilde{a}_3 \leq \tilde{a}_8) \}.$$

Then we consider the substitution:

$$a_9\sigma = a_1, \quad a_4\sigma = a_{11}, \quad \tilde{a}_8\sigma = \tilde{a}_3, \quad \tilde{a}_4\sigma = \tilde{a}_{11}, \quad a_8\sigma = X, \quad \tilde{a}_9\sigma = Y$$

Then we solve the following requirements:

1) $\epsilon a_0 \leq \tilde{a}_0,\ a_1\tilde{a}_1 \leq a_0,\ a_3\tilde{a}_3 \leq \tilde{a}_1,\ Ya_{11} \leq \tilde{a}_{11} \mid\!\approx a_3a_{11} \leq \tilde{a}_{11}$

2) $\epsilon a_0 \leq \tilde{a}_0,\ a_1\tilde{a}_1 \leq a_0,\ a_3\tilde{a}_3 \leq \tilde{a}_1,\ Ya_{11} \leq \tilde{a}_{11} \mid\!\approx X\tilde{a}_3 \leq \tilde{a}_0$

3) $\epsilon a_0 \leq \tilde{a}_0,\ a_1\tilde{a}_1 \leq a_0,\ a_3\tilde{a}_3 \leq \tilde{a}_1,\ Ya_{11} \leq \tilde{a}_{11} \mid\!\approx a_1Y \leq X$

From 1) we directly deduce $Y = a_3$ and also $a_4\gamma = \{\, a_{11} \,\}$. The requirement in 1) is the one of the position a_4 and in order to verify it we use the assertion $a_3a_{11} \leq \tilde{a}_{11}$ of position a_{11}. From 3) we deduce a trivial solution for X that is $X = a_1a_3$ and also that $a_9\gamma = \emptyset$. Requirement 2) is verified because we have:

$$\dfrac{\dfrac{\dfrac{a_1 \leq a_1 \quad a_3\tilde{a}_3 \leq \tilde{a}_1}{a_1a_3\tilde{a}_3 \leq a_1\tilde{a}_1}\,F \qquad a_1\tilde{a}_1 \leq a_0}{a_1a_3\tilde{a}_3 \leq a_0}\,T \qquad \epsilon a_0 \leq \tilde{a}_0}{a_1a_3\tilde{a}_3 \leq \tilde{a}_0}\,T$$

and then we deduce $a_8\gamma = \{\, a_0, a_1, a_3 \,\}$ since $\epsilon a_0 \leq \tilde{a}_0$, $a_1\tilde{a}_1 \leq a_0$, $a_3\tilde{a}_3 \leq \tilde{a}_1$ are the respective assertions of a_0, a_1, a_3.

In order to verify the conditions $(C2)$ to $(C5)$, let us consider the following table:

$(s, \langle\, u, v \,\rangle)$	$P_\beta(\{\, u, v \,\}\uparrow)$	$P_\alpha(s\uparrow)$	$\Sigma_\beta(\{\, u, v \,\}\uparrow)$	$\Sigma_\alpha(s\uparrow)$
$(s_1, \langle\, a_2, a_{10} \,\rangle)$	a_8, a_9	a_0, a_1, a_3	$a_8, \tilde{a}_8, a_9, \tilde{a}_9$	$a_0, \tilde{a}_0, a_1, \tilde{a}_1, a_3, \tilde{a}_3$
$(s_2, \langle\, a_{12}, a_5 \,\rangle)$	a_4, a_8, a_9	a_0, a_1, a_3, a_{11}	$a_4, \tilde{a}_4, a_8, \tilde{a}_8, a_9, \tilde{a}_9$	$\tilde{a}_0, a_1, \tilde{a}_1, a_3, \tilde{a}_3, a_{11}, \tilde{a}_{11}$
$(s_3, \langle\, a_7, a_{14} \,\rangle)$	a_8, a_9	a_0, a_1, a_3	$a_8, \tilde{a}_8$	$a_0, \tilde{a}_0, a_1, \tilde{a}_1, a_3, \tilde{a}_3$
$(s_4, \langle\, a_2, a_{10} \,\rangle)$	a_8, a_9	a_0, a_1, a_3	$a_8, \tilde{a}_8, a_9, \tilde{a}_9$	$a_0, \tilde{a}_0, a_1, \tilde{a}_1, a_3, \tilde{a}_3$
$(s_5, \langle\, a_6, a_{13} \,\rangle)$	a_4, a_8, a_9	a_0, a_1, a_3, a_{11}	$a_4, \tilde{a}_4, a_8, \tilde{a}_8, a_9, \tilde{a}_9$	$\tilde{a}_0, a_1, \tilde{a}_1, a_3, \tilde{a}_3, a_{11}, \tilde{a}_{11}$
$(s_6, \langle\, a_7, a_{14} \,\rangle)$	a_8, a_9	a_0, a_1, a_3	$a_8, \tilde{a}_8$	$a_0, \tilde{a}_0, a_1, \tilde{a}_1, a_3, \tilde{a}_3$

We have $a_9\gamma = \emptyset \subseteq P_\alpha(s\uparrow)$ for all paths $s \in \{\, s_1, s_2, s_4, s_5, s_6 \,\}$ and $a_8\gamma = \{\, a_0, a_1, a_3 \,\} \subseteq P_\alpha(s\uparrow)$ for all paths $s \in \{\, s_1 \ldots s_6 \,\}$. Moreover, for all paths $s \in \{\, s_2, s_5 \,\}$, $a_4\gamma = \{\, a_{11} \,\} \subseteq P_\alpha(s\uparrow)$. Then the condition $(C2)$ is verified. In addition, for all paths $s \in \{\, s_1, s_2, s_4, s_5 \,\}$ we have $a_9\sigma = a_1 \in \mathcal{L}_\alpha(s\uparrow)$ and $\tilde{a}_9\sigma = a_3 \in \mathcal{L}_\alpha(s\uparrow)$ for all paths $s \in \{\, s_1, s_2, s_3, s_4, s_5, s_6 \,\}$ we have $a_8\sigma = a_1a_3 \in \mathcal{L}_\alpha(s\uparrow)$ and $\tilde{a}_8\sigma = \tilde{a}_3 \in \mathcal{L}_\alpha(s\uparrow)$, and for all paths $s \in \{\, s_2, s_5 \,\}$ we have $a_4\sigma = a_{11} \in \mathcal{L}_\alpha(s\uparrow)$ and $\tilde{a}_4\sigma = \tilde{a}_{11} \in \mathcal{L}_\alpha(s\uparrow)$.

The last thing to do is to compute the reduction relation $\lhd$ that is obtained by the transitive closure of the domination relation $\ll$, the instantiation relation $\sqsubset$ and the deduction relation $\sqsubset$. The instantiation relation induced by σ is

$$a_1 \sqsubset a_9, a_3 \sqsubset a_9, a_1 \sqsubset a_8, a_{11} \sqsubset a_4$$

and the deduction relation induced by γ is

$$a_0 \sqsubset a_8, a_1 \sqsubset a_8, a_3 \sqsubset a_8, a_{11} \sqsubset a_4.$$

The reduction relation $\lhd$ is represented in Figure 5. As the graph is acyclic, A is valid in MILL.

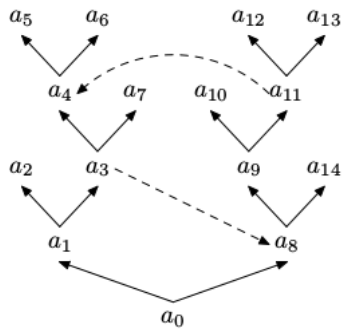


Figure 5: Reduction order for $(\mathrm{P} * ((\mathrm{Q} \multimap\!\!* \mathrm{R}) * \mathrm{S}) \multimap\!\!* ((\mathrm{P} * (\mathrm{Q} \multimap\!\!* \mathrm{R})) * \mathrm{S})$

4 Properties of the Characterization

In this section we prove the soundness and completeness of characterization given in Definition 12. The completeness is proved by showing that any formula provable in the LMILL sequent calculus is also provable according to the connection-based characterization (CMILL-provable).

Definition 13 (Complete reduction)**.** *Let* A *be an indexed formula and C be a cover of* A*, a reduction $(\mathcal{S}_i)_{1\leq i\leq n}$ in* A *is* complete *for C if C is a cover of $\mathcal{S}_n$.*

Definition 14 (Proper reduction)**.** *Let* A *be an indexed formula and σ be a substitution for* A*, a reduction $(\mathcal{S}_i)_{1\leq i\leq n}$ is σ-*proper *iff*

1. $\forall S \in (\mathcal{S}_i)_{1\leq i\leq n}, \forall s \in S, \mathcal{K}_\alpha(s\uparrow)\sigma \models\!\!\approx \mathcal{K}_\beta(s\uparrow)\sigma$ *and*
2. $\forall S \in (\mathcal{S}_i)_{0\leq i\leq n}, \forall s \in S, \forall x \in \Sigma_\beta(s\uparrow), x\sigma \in \mathcal{L}_\alpha(s\uparrow)$.

Definition 15 (Realization)**.** *Let* A *be an indexed formula and s be a path in* A*. An* interpretation *of s in a resource model $\mathcal{R} = \langle (M, \sqsubseteq, \cdot, e), \models, [\![-]\!] \rangle$ is a function $\|-\| : \Sigma_\alpha(s\uparrow) \to M$ that can be extended to labels $\mathcal{L}_\alpha(s\uparrow)$ with $\|\epsilon\| = e$ and $\|xy\| = \|x\| \cdot \|y\|$.*
Given a substitution σ for A*, we denote $\|-\|_\sigma$ the composed function $\|-\| \circ \sigma$ from the set $\mathcal{L}_{\alpha\beta}(s\uparrow)$ of labels of s to the set of worlds M of $\mathcal{R}$.*
A realization *of s is a couple $(\|-\|, \sigma)$ such that:*

1. *For all assertions $x \leq y \in \mathcal{K}_\alpha(s\uparrow)$, $\|x\|_\sigma \sqsubseteq \|y\|_\sigma$.*
2. *For all positions $u \in s$ such that $lsf(u) = \mathrm{A}^1 : x$, $\|x\|_\sigma \models \mathrm{A}$.*
3. *For all positions $u \in s$ such that $lsf(u) = \mathrm{A}^0 : x$, $\|x\|_\sigma \not\models \mathrm{A}$.*

A path is realizable *if there exists a realization of s in a model $\mathcal{R}$. A configuration is* realizable *if at least one of its paths is realizable.*

Lemma 1. *Let s be a path in an indexed formula* A*, $(\|-\|, \sigma)$ be a realization of s in a model $\mathcal{R} = (M, e, \cdot, \sqsubseteq, \models)$ and $K \subseteq \mathcal{K}_\alpha(s\uparrow)$ a subset of assertions associated to s. If $K\sigma \models\!\!\approx (x \leq y)\sigma$ then $\|x\|_\sigma \sqsubseteq \|y\|_\sigma$.*

Proof. By definition of a realization, for any assertion $x' \leq y'$ of K we have $\|x'\|_\sigma \sqsubseteq \|y'\|_\sigma$. By hypothesis, under σ, the constraint $x \leq y$ is deduced (in the K-deduction system) from assertions of K by rules expressing reflexivity and transitivity of $\leq$ and also compatibility of label composition with $\leq$. Moreover, by definition of a preorder, $\sqsubseteq$ is reflexive and transitive and by definition of a MILL-model, world composition is compatible with $\sqsubseteq$ in $\mathcal{R}$. Consequently, the rules of the K-deduction system transfer the notion of realizability from premisses to conclusion. □

Lemma 2. *Let* A *be an indexed formula,* σ *be a substitution for* A *and* $(\mathcal{S}_i)_{1\leq i\leq n}$ *be a* σ*-proper reduction for* A*, if* $\mathcal{S}_i$ *is* σ*-realizable then* $\mathcal{S}_{i+1}$ *is* σ*-realizable.*

Proof. As the configuration $\mathcal{S}_i$ is realizable under σ, it contains a path s that is realizable under σ in a model $(M, e, \cdot, \sqsubseteq, \models)$ for a interpretation $\|-\|$. Let us suppose that $\mathcal{S}_{i+1}$ is obtained by a reduction of a position u in a path of $\mathcal{S}$. If $u \notin s$ then $\mathcal{S}_{i+1}$ remains realizable under σ because it always contains the path s. Otherwise, we proceed by case analysis depending on the principal connective of $lsf(u) = \mathrm{A}^{\mathrm{pol}} : x$ and show that $\mathcal{S}_{i+1}$ remains realizable under σ:

- Case $(\mathrm{B} * \mathrm{C})^1 : x$
  The path s is reduced into s' by replacing the position u by its children positions $[u]_1$ and $[u]_2$ such that $lsf([u]_1) = \mathrm{B}^1 : u$ and $lsf([u]_2) = \mathrm{C}^1 : \tilde{u}$. Then $\Sigma_\alpha(s'\uparrow) = \Sigma_\alpha(s\uparrow) \cup \{\, u, \tilde{u} \,\}$ and $\mathcal{K}_\alpha(s'\uparrow) = \mathcal{K}_\alpha(s\uparrow) \cup \{\, u\tilde{u} \leq x \,\}$. By hypothesis, $\|-\|_\sigma$ is a realization of s and then $\|x\|_\sigma \models \mathrm{B} * \mathrm{C}$. Thus, by definition of $\models$, there exist two worlds $m, n \in M$ such that $m \cdot n \sqsubseteq \|x\|_\sigma$, $m \models \mathrm{B}$ and $n \models \mathrm{C}$. We then extend $\|-\|$ to u and $\tilde{u}$ by defining $\|u\| = m$ and $\|\tilde{u}\| = n$ to obtain $\|u\|_\sigma \models \mathrm{B}$, $\|\tilde{u}\|_\sigma \models \mathrm{C}$ and $\|u\tilde{u}\|_\sigma = \|u\|_\sigma \cdot \|\tilde{u}\|_\sigma \sqsubseteq \|x\|_\sigma$. Consequently s' is realizable under σ.

- Case $(\mathrm{B} * \mathrm{C})^0 : x$
  The path is reduced into two paths s' and s'' such that $\Sigma_\beta(s'\uparrow) = \Sigma_\beta(s\uparrow) \cup \{\, u \,\}$, $\Sigma_\beta(s''\uparrow) = \Sigma_\beta(s\uparrow) \cup \{\, \tilde{u} \,\}$ and $\mathcal{K}_\beta(s'\uparrow) = \mathcal{K}_\beta(s''\uparrow) = \mathcal{K}_\beta(s\uparrow) \cup \{\, u\tilde{u} \leq x \,\}$. By hypothesis the reduction $(\mathcal{S}_i)_{1\leq i\leq n}$ is σ-proper and we have $\mathcal{K}_\alpha(s'\uparrow)\sigma \mathrel{|\!\approx} \mathcal{K}_\beta(s'\uparrow)\sigma$ and $\mathcal{K}_\alpha(s''\uparrow)\sigma \mathrel{|\!\approx} \mathcal{K}_\beta(s''\uparrow)\sigma$. In particular $\mathcal{K}_\alpha(s\uparrow) = \mathcal{K}_\alpha(s'\uparrow) = \mathcal{K}_\alpha(s''\uparrow)$, $\mathcal{K}_\beta(s'\uparrow) = \mathcal{K}_\beta(s''\uparrow)$ and $u\tilde{u} \leq x \in \mathcal{K}_\beta(s'\uparrow)$ entail $\mathcal{K}_\alpha(s\uparrow)\sigma \mathrel{|\!\approx} (u\tilde{u} \leq x)\sigma$. As $\|-\|_\sigma$ is a realization of s, we have $\|x\|_\sigma \not\models \mathrm{B} * \mathrm{C}$ and from Lemma 1 we deduce $\|u\|_\sigma \cdot \|\tilde{u}\|_\sigma \sqsubseteq \|x\|_\sigma$. Then, by definition of $\models$, for all worlds $m, n \in M$ such that $m \cdot n \sqsubseteq \|x\|_\sigma$, we have either $m \not\models \mathrm{B}$ or $n \not\models \mathrm{C}$. In particular, we have either $\|u\|_\sigma \not\models \mathrm{B}$ or $\|\tilde{u}\|_\sigma \not\models \mathrm{C}$. Consequently, either s' is realizable under σ or s'' is realizable under σ.

- The other cases are similar.

□

Lemma 3. *Let* A *be an indexed formula and* σ *be a substitution for* A*. If a path* s *is complementary under* σ *then it is not realizable under* σ*.*

Proof. Let us suppose that s contains a connection $\langle\, u, v \,\rangle$ such that $f(u) = f(v)$, $pol(u) = 1$, $pol(v) = 0$ and $\mathcal{K}_\alpha(s\uparrow)\sigma \mathrel{|\!\approx} slab(u)\sigma \leq slab(v)\sigma$. If s is realizable under σ for an interpretation $\|-\|$ in a model $\mathcal{R}$ then $\|slab(u)\|_\sigma \models f(u)$, $\|slab(v)\|_\sigma \not\models f(u)$ and $\|slab(u)\|_\sigma \sqsubseteq \|slab(v)\|_\sigma$, which is contradictory because by Kripke monotonicity $\|slab(u)\|_\sigma \sqsubseteq \|slab(v)\|_\sigma$ and $\|slab(u)\|_\sigma \models f(u)$ entail $\|slab(v)\|_\sigma \models f(u)$. The case of a 1-connection is similar. □

Theorem 2 (Soundness)**.** *If a formula* A *is CMILL-provable then it is valid.*

Proof. As A is provable, there is a cover C of the set of atomic paths of A, a substitution σ and a certification γ for A satisfying the conditions of Definition 12.
Let us suppose that A is not valid. Then, there exists a model $\mathcal{R} = (M, e, \cdot, \sqsubseteq, \models)$ such that $e \not\models \mathrm{A}$. The initial configuration $\mathcal{S}_1 = \{\, \{\, a_0 \,\} \,\}$ is then trivially realizable under σ by considering the interpretation $\|-\|$ the domain of which is empty. It is easy to show that conditions $(C1)$ to $(C5)$ entail the existence of a reduction $(\mathcal{S}_i)_{1 \leq i \leq n}$ from $\mathcal{S}_1$ that is complete for C, σ-proper and such that all paths of $\mathcal{S}_n$ contain at least a connection of C. As $\mathcal{S}_1$ is realizable under σ, Lemma 2 entails that the configuration $\mathcal{S}_n$ is also realizable under σ. But then, by Lemma 3, we deduce that $\mathcal{S}_n$ cannot be complementary, which is a contradiction. Therefore, A is valid. □

Let us now consider the question of completeness of this characterization.

Theorem 3 (Completeness)**.** *If a formula* A *is valid then* A *is CMILL-provable.*

Proof. From the sound and completeness of MILL-models, it is sufficient to prove that if A is LMILL-provable then A is CMILL-provable (provable by the connection characterization). The proof is by induction on a LMILL-proof of A, knowing that a sequent $\Gamma \vdash \mathrm{A}$ is provable in LMILL if and only if the formula $\Phi_\Gamma \multimap\!\!* \, \mathrm{A}$ is provable in LMILL, where Φ_Γ is the formula obtained by replacing each comma in the context Γ with multiplicative conjunction $*$.

- Case ax: the axiom $\mathrm{A} \vdash \mathrm{A}$ corresponds to the formula $\mathrm{A} \multimap\!\!* \, \mathrm{A}$ which is trivially CMILL-provable.

- Case $\multimap\!\!*_R$: By induction hypothesis we suppose that the sequent $\Gamma, \mathrm{A} \vdash \mathrm{B}$ is provable and we show that the sequent $\Gamma \vdash \mathrm{A} \multimap\!\!* \, \mathrm{B}$ is also provable. If $\Gamma, \mathrm{A} \vdash \mathrm{B}$ is CMILL-provable an atomic reduction $\mathcal{R}_1 = (\mathcal{S}_i)_{1 \leq i \leq n}$ of $((\Phi_\Gamma * \mathrm{A}) \multimap\!\!* \, \mathrm{B})$, a cover C of $\mathcal{S}_n$, a substitution σ and a certification γ for A that satisfy the conditions of Definition 12.

  From the atomic reduction $\mathcal{R}_1$ for $((\Phi_\Gamma * \mathrm{A}) \multimap\!\!* \mathrm{B})$ we can build an atomic reduction $\mathcal{R}_2$ for $(\Phi_\Gamma \multimap\!\!* (\mathrm{A} \multimap\!\!* \mathrm{B}))$. On the left-hand side of the next figure, we describe the first steps of the reduction $\mathcal{R}_1$ and, on the right-hand side, we describe the first steps of the corresponding reduction $\mathcal{R}_2$. We represent here a path s as a set of signed formulae and not as a set of positions, namely we have $lsf(s) = \{\, lsf(u) \mid u \in s \,\}$.

$$
\begin{array}{c|c}
\{\, ((\Phi_\Gamma * \mathrm{A}) \multimap\!\!* \mathrm{B})^0 : \epsilon \,\} & \{\, (\Phi_\Gamma \multimap\!\!* (\mathrm{A} \multimap\!\!* \mathrm{B}))^0 : \epsilon \,\} \\
| & | \\
\{\, (\Phi_\Gamma * \mathrm{A})^1 : a_0, \mathrm{B}^0 : \tilde{a}_0 \,\} & \{\, {\Phi_\Gamma}^1 : a_0, (\mathrm{A} \multimap\!\!* \mathrm{B})^0 : \tilde{a}_0 \,\} \\
| & | \\
\{\, {\Phi_\Gamma}^1 : a_1, \mathrm{A}^1 : \tilde{a}_1, \mathrm{B}^0 : \tilde{a}_0 \,\} & \{\, {\Phi_\Gamma}^1 : a_0, \mathrm{A}^1 : a_i, \mathrm{B}^0 : \tilde{a}_i \,\} \\
| & | \\
\vdots & \vdots \\
\mathcal{R}_1 & \mathcal{R}_2
\end{array}
$$

  We observe that the first reduction steps in $\mathcal{R}_1$ and $\mathcal{R}_2$ lead to paths containing the same signed formulae, modulo a renaming of a_1 into a_0, of $\tilde{a}_1$ into a_i and of $\tilde{a}_0$ into $\tilde{a}_i$. Consequently, modulo the renaming, we can reduce $\mathcal{R}_2$ by applying exactly the same reduction steps than for $\mathcal{R}_1$. Then, after the two first steps previously described, $\mathcal{R}_1$ and $\mathcal{R}_2$ introduce the same signed formulae and then the same constraints (assertions and requirements) between labels, modulo renaming.
  Moreover, the assertions $\{\, a_0\tilde{a}_0 \leq, a_1\tilde{a}_1 \leq \tilde{a}_0 \,\}$ introduced in the two first steps of $\mathcal{R}_1$ entail relations between labels $\tilde{a}_0, a_1, \tilde{a}_1$ of $\mathcal{R}_1$ weaker than the ones between labels $a_0, a_i, \tilde{a}_i$ in

$$\frac{}{\Gamma, x \leq y, \mathrm{A} : x \vdash \mathrm{A} : y, \Delta}\, id \qquad \frac{}{\Gamma, \epsilon \leq x \vdash x : 1, \Delta}\, 1_R \qquad \frac{\Gamma, \epsilon \leq x \vdash \Delta}{\Gamma, 1 : x \vdash \Delta}\, 1_L$$

$$\frac{\Gamma, ab \leq x, \mathrm{A} : a, \mathrm{B} : b \vdash \Delta}{\Gamma, \mathrm{A} * \mathrm{B} : x \vdash \Delta}\, *_L \qquad \frac{yz \leq x, \Gamma \vdash \mathrm{A} : y, \Delta \qquad yz \leq x, \Gamma \vdash \mathrm{B} : z, \Delta}{\Gamma, yz \leq x \vdash \mathrm{A} * \mathrm{B} : x, \Delta}\, *_R$$

$$\frac{\Gamma, xy \leq z \vdash \mathrm{A} : y, \Delta \qquad \Gamma, xy \leq z, \mathrm{B} : z \vdash \Delta}{\Gamma, xy \leq z, \mathrm{A} \multimap \mathrm{B} : x \vdash \Delta}\, \multimap_L \qquad \frac{\Gamma, \mathrm{A} : a, xa \leq b \vdash \mathrm{B} : b, \Delta}{\Gamma \vdash \mathrm{A} \multimap \mathrm{B} : x, \Delta}\, \multimap_R$$

$$\frac{\Gamma, x \leq x \vdash \Delta}{\Gamma \vdash \Delta}\, R \qquad \frac{\Gamma, x \leq z, x \leq y, y \leq z \vdash \Delta}{\Gamma, x \leq y, y \leq z \vdash \Delta}\, T \qquad \frac{\Gamma, xx' \leq yy', x \leq x', y \leq y' \vdash \Delta}{\Gamma, x \leq y, x' \leq y' \vdash \Delta}\, F$$

Side conditions:
– In $*_L$ and $\multimap_R$, the constants a and b do not occur in the conclusion.
– In R the label x must already occur in the conclusion.

Figure 6: Labelled Sequent Calculus GMILL

$\mathcal{R}_2$ by assertions $\{ a_0\tilde{a}_0 \leq, \tilde{a}_0 a_i \leq \tilde{a}_i \}$ introduced in the two first steps of $\mathcal{R}_2$. In fact, $\mathcal{R}_1$ imposes $slab(\Phi_\Gamma)slab(\mathrm{A}) \leq slab(\mathrm{B})$ while $\mathcal{R}_2$ imposes $slab(\Phi_\Gamma)slab(\mathrm{A}) = slab(\mathrm{B})$. Consequently, as $\mathcal{R}_1$ leads to a set of atomic paths satisfying the conditions of Definition 12, it is the same for $\mathcal{R}_2$ by induction hypothesis.

- Case $*_L$: immediate by the translation Φ, because $\Phi_{\Gamma,\mathrm{A},\mathrm{B}} = \Phi_{\Gamma,\mathrm{A}*\mathrm{B}}$.
- The other cases are similar.

□

5 A Labelled Sequent Calculus for MILL

From the previous characterization we can derive a sound and complete labelled sequent calculus GMILL[3] for MILL. Soundness and completeness are easy consequences of Theorem 2 and Theorem 3.

Labels and constraints for GMILL are defined similarly as in Section 3.1 except that GMILL does not make use of variables. The sequent calculus GMILL deals with sequents of the form $\Gamma \vdash \Delta$ where Γ and Δ are multisets containing labelled formulas, Γ being allowed to also contain constraints. Labelled formulas are pairs (A, x), written $\mathrm{A} : x$, where A is a formula and x is a label. Label constraints occurring in Γ are called assertions. We denote Γ_r the restriction of Γ to its constraints. GMILL does not have explicit requirements. Instead, the rules $*_R$ and $\multimap_L$ are required to have a specific constraint (which in the connection-based characterization

[3] The G in GMILL is reminiscent of the fact that labels and label-constraints can be viewed as a graphical structure we usually call a *resource-graph* in related works.

$$\dfrac{}{\ldots, a_1 \leq a_1, \ldots, \mathrm{P} : a_1, \ldots \vdash \ldots, \mathrm{P} : a_1}$$
$$\Pi_3$$

$$\dfrac{}{\ldots, a_3 \leq a_3, \ldots, \mathrm{Q} \multimap \mathrm{R} : a_3, \ldots \vdash \ldots, \mathrm{Q} \multimap \mathrm{R} : a_3}$$
$$\Pi_4$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi_3 \qquad \Pi_4}{a_1a_3 \leq a_1a_3, a_3 \leq a_3, \ldots, a_1 \leq a_1, \ldots, \mathrm{P} : a_1, \mathrm{Q} \multimap \mathrm{R} : a_3, \ldots \vdash \ldots, \mathrm{P} * (\mathrm{Q} \multimap \mathrm{R}) : a_1a_3}\,*_R}{a_3 \leq a_3, \ldots, a_1 \leq a_1, \ldots, \mathrm{P} : a_1, \mathrm{Q} \multimap \mathrm{R} : a_3, \ldots \vdash \ldots, \mathrm{P} * (\mathrm{Q} \multimap \mathrm{R}) : a_1a_3}\,F}{a_1a_3\tilde{a}_3 \leq \tilde{a}_0, \ldots, \mathrm{P} : a_1, \mathrm{Q} \multimap \mathrm{R} : a_3, \ldots \vdash \ldots, \mathrm{P} * (\mathrm{Q} \multimap \mathrm{R}) : a_1a_3}\,R}{\Pi_1}$$

$$\dfrac{\dfrac{}{\tilde{a}_3 \leq \tilde{a}_3, \ldots, \mathrm{S} : \tilde{a}_3 \vdash \ldots, \mathrm{S} : \tilde{a}_3}}{a_1a_3\tilde{a}_3 \leq \tilde{a}_0, a_1a_3\tilde{a}_3 \leq a_1\tilde{a}_1, a_1 \leq a_1, a_3\tilde{a}_3 \leq \tilde{a}_1, a_1\tilde{a}_1 \leq \tilde{a}_0, \epsilon a_0 \leq \tilde{a}_0, \ldots, \mathrm{S} : \tilde{a}_3 \vdash \ldots, \mathrm{S} : \tilde{a}_3}\,R$$
$$\Pi_2$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1 \qquad \Pi_2}{a_1a_3\tilde{a}_3 \leq \tilde{a}_0, \ldots, \mathrm{P} : a_1, \mathrm{Q} \multimap \mathrm{R} : a_3, \mathrm{S} : \tilde{a}_3 \vdash ((\mathrm{P} * (\mathrm{Q} \multimap \mathrm{R})) * \mathrm{S}) : \tilde{a}_0}\,*_R}{a_1a_3\tilde{a}_3 \leq a_1\tilde{a}_1, \ldots, a_1\tilde{a}_1 \leq \tilde{a}_0, \ldots, \mathrm{P} : a_1, \mathrm{Q} \multimap \mathrm{R} : a_3, \mathrm{S} : \tilde{a}_3 \vdash ((\mathrm{P} * (\mathrm{Q} \multimap \mathrm{R})) * \mathrm{S}) : \tilde{a}_0}\,T}{a_1 \leq a_1, a_3\tilde{a}_3 \leq \tilde{a}_1, \ldots, \mathrm{P} : a_1, \mathrm{Q} \multimap \mathrm{R} : a_3, \mathrm{S} : \tilde{a}_3 \vdash ((\mathrm{P} * (\mathrm{Q} \multimap \mathrm{R})) * \mathrm{S}) : \tilde{a}_0}\,F}{a_3\tilde{a}_3 \leq \tilde{a}_1, a_1\tilde{a}_1 \leq \tilde{a}_0, \epsilon a_0 \leq \tilde{a}_0, \mathrm{P} : a_1, \mathrm{Q} \multimap \mathrm{R} : a_3, \mathrm{S} : \tilde{a}_3 \vdash ((\mathrm{P} * (\mathrm{Q} \multimap \mathrm{R})) * \mathrm{S}) : \tilde{a}_0}\,R}{a_1\tilde{a}_1 \leq \tilde{a}_0, \epsilon a_0 \leq \tilde{a}_0, \mathrm{P} : a_1, ((\mathrm{Q} \multimap \mathrm{R}) * \mathrm{S}) : \tilde{a}_1 \vdash ((\mathrm{P} * (\mathrm{Q} \multimap \mathrm{R})) * \mathrm{S}) : \tilde{a}_0}}{\epsilon a_0 \leq \tilde{a}_0, (\mathrm{P} * ((\mathrm{Q} \multimap \mathrm{R}) * \mathrm{S}) : a_0 \vdash ((\mathrm{P} * (\mathrm{Q} \multimap \mathrm{R})) * \mathrm{S}) : \tilde{a}_0}}{\vdash (\mathrm{P} * ((\mathrm{Q} \multimap \mathrm{R}) * \mathrm{S})) \multimap ((\mathrm{P} * (\mathrm{Q} \multimap \mathrm{R})) * \mathrm{S}) : \epsilon}$$

Figure 7: GMILL-proof of $(\mathrm{P} * ((\mathrm{Q} \multimap \mathrm{R}) * \mathrm{S})) \multimap ((\mathrm{P} * (\mathrm{Q} \multimap \mathrm{R})) * \mathrm{S})$

corresponds to a requirement) occuring in Γ_{r} for the rules to be applicable. The rules of the GMILL calculus are given in Figure 6.

From the proof of Theorem 3, one can derive a translation of LMILL-proofs into GMILL-proofs so that for any LMILL-proof the corresponding GMILL-proof applies the same rules in the same order. Therefore, since LMILL does not allow contraction, GMILL has no need for it too. Moreover, conditions $(C1)$ to $(C5)$ given in Definition 12 imply that whenever $*_R$ and $\multimap_L$ need to be applied in GMILL, Γ_{r} contains enough assertions to make the rule applicable. Therefore we have the following results.

Theorem 4. *If a formula* A *has a proof in LMILL then it is has a proof in GMILL that follows the same rule application strategy.*

Theorem 5. *A formula* A *is provable in LMILL iff it is provable in GMILL.*

Figure 5 illustrates how GMILL works by giving an example of a derivation in the GMILL

calculus for the formula $(P * ((Q \multimap R) * S)) \multimap ((P * (Q \multimap R)) * S)$, which is exactly the one prescribed by the reduction ordering we computed in the running example of Section 3.4.

6 Future Work

From this connection-based characterization of validity in MILL we will consider different perspectives. First we aim at defining a connection method for MILL from such a characterization that mainly corresponds to the definition and implementation of an algorithm for solving our constraints. A complementary question consist in studying how our results can be adapted or refined to deal with other fragments of Intuitionistic Linear Logic and mainly first-order ones including quantifications.

The question of reconstruction of proofs in the MILL sequent calculus from our connection calculus for MILL with labels and constraints has also to be explored w.r.t. existing techniques [11]. Moreover, taking into account the relationships in MLL between some connection-based characterizations and proof-nets [1], we aim at studying similar relationships for MILL and propose new proof methods based on proof-net construction.

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