Tree-sequent Calculi and Decision Procedures for Intuitionistic Modal Logics

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Abstract. In this paper we define label-free sequent calculi for the intuitionistic modal logics obtained from the combinations of the axioms T, B, 4 and 5. These calculi are based on a multi-contextual sequent structure, called Tree-sequent, that allows us to define such calculi for such intuitionistic modal logics. From the calculi defined for the IK, IT, IB4 and ITB logics, we also provide new decision procedures and alternative syntactic proofs of decidability.

1 Introduction

In this paper, we are mainly interested in proof theory and sequent proof systems for intuitionistic modal logics obtained from the combinations of the axioms T, B, 4 and 5 [27]. Intuitionistic modal logics have been studied in different perspectives [10, 11, 13, 22, 24, 28] with some applications in computer science, for instance in the formal verification of computer hardware [8] and in the definition of some programming languages [7, 19]. Moreover modal proofs have been also studied in the perspective of distributed programs [18].

In the context of proof theory for modal logics we notice that traditional sequent proof systems cannot easily meet basic and important requirements like cut-elimination and subformula properties and thus some generalizations have been recently proposed for classical modal logics [21]. Moreover there are not so many works devoted to sequent calculi for intuitionistic modal logics, for instance obtained from the combinations of the axioms T, B, 4 and 5 [2, 6, 22, 27]. The main approach for designing sequent calculi for these intuitionistic modal logics is based on labels that explicitly integrate some semantic information, like the accessibility relation, into the proof systems [27]. The calculi that are obtained in this way do not satisfy for instance the subformula property, mainly because of labels that are external to the logic language.

In this work we focus on the definition of label-free sequent calculi, for the above mentioned intuitionistic logics, with good properties like cut-elimination and subformula properties [29]. Such based-on sequent calculi are important in order to design decision procedures and to define methods and tools for proof-search. For classical modal logics and their variants there exist various kinds of calculi like display calculi [17, 30] or labelled sequent calculi [20]. Recently label-free calculi have been developed by using so-called nested sequents [12], that can be seen as a generalization of the hypersequent structure [1]. Such sequents have been defined for various logics, including classical modal logics, under the name of deep sequent [3, 4, 16] or of tree-hypersequent [25] and provide label-free sequent calculi that satisfy the cut-elimination and subformula properties. But such structures cannot be used for intuitionistic modal logics, in which the disjunction property is satisfied, and then other structures are needed for these logics. Thus, in order to design proof-theoretic systems for intuitionistic modal logics that stay inside the modal language and that are label-free, we consider a new multi-contextual sequent structure, called Tree-sequent (or T-sequent) that can be seen as a specific variant of deep sequent properly defined for these logics. It generalizes another multi-contextual sequent structure that allows us to define label-free natural deduction and sequent calculi for the particular intuitionistic modal logic IS5 [15]. The structure of Tree-sequent has been used for defining natural deduction systems for intuitionistic modal logics [14] and in this paper we focus on its use for defining label-free sequent calculi for intuitionistic modal logics obtained from the combinations of the axioms T, B, 4 and 5 and then for deriving new related decision procedures and proof-search methods for some of them.

After a presentation of syntax and semantics of intuitionistic modal logics we define the structure of Tree-sequent and remind how natural deduction systems for such logics can be defined with this structure. Then we define based-on T-sequent calculi in the standard Kleene-style with contraction absorbed in all the left rules (for more details see [29]). In order to prove cut-elimination, we first consider the corresponding natural deduction systems [14] and their normalisation property, following an approach similar to that of Prawitz [26]. But we also provide a direct proof of cut-elimination by showing the admissibility of the cut rule in the sequent calculi. Then we deduce from the cut-elimination property the subformula property and another property, called depth property, about the depth of the T-sequents in a cut free proof. It means that there is a value that bounds the depth of the T-sequents appearing in a cut-free derivation.

A complementary contribution consists in defining new decision procedures based on our new sequent calculi for the logics IK, IT, IB4 and ITB. Our approach is based on a notion of redundancy in cut-free derivations so that any valid T-sequent has an irredundant derivation as a proof. Then, by using the subformula property and also the depth property, we show that there is no infinite proof which is not irredundant and then we define new decision procedures that provide alternative proofs the decidability of IK, IT, IB4 and ITB through proof-search. We restrict ourselves to the above logics because the calculi for the other ones do not satisfy the depth property that is required for termination in our approach.

Further work will be devoted to the design of label-free sequent calculi for other intuitionistic modal logics with appropriate structures and also the study of the decidability of these logics by using such sequent calculi.

2 Intuitionistic modal logics

Intuitionistic modal logics (IML) are the logics we obtain by replacing in classical modal logics the classical reasoning principles with the intuitionistic ones [27]. Let us remind that these logics have applications in computer science, like for formal verification [8] and for the definition of programming languages [7, 19].

2.1 Syntax and semantics

The language of intuitionistic modal logics is obtained from the language of propositional logic by adding two unary connectives $\Box$ and $\Diamond$. More precisely, the set of formulae, denoted Form, is inductively defined from a set of propositional variables, denoted Prop, the constant $\bot$ denoting absurdity, and using the logical connectives $\wedge$, $\vee$, $\supset$, $\Box$ and $\Diamond$. In other words, the formulæ are defined using the following grammar:

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid \Box A \mid \Diamond A$$

We use $p, q, r \ldots$ as metavariables for propositional variables; and $A, B, C, \ldots$ as metavariables for generic formulæ. The negation, denoted $\neg$, can be defined by using $\bot$ and the connective $\supset$ as follows: $\neg A \triangleq A \supset \bot$. The constant true is defined by $\top \triangleq \bot \supset \bot$.

Let us note that the interdefinability between $\Box$ and $\Diamond$ given by $\Diamond A \triangleq \neg\Box\neg A$ we have in classical intuitionistic logics is not present in intuitionistic modal logics. That is similar to the fact that $\forall$ and $\exists$ are independent in intuitionistic first-order logic.

Let us first consider the intuitionistic modal logic IK. A Hilbert axiomatic system for this logic is defined as follows [27]:

- All substitution instances of theorems of IPL;
- $\Box(A \supset B) \supset (\Box A \supset \Box B)$;
- $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$;
- $\Diamond\bot \supset \bot$;
- $\Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$;
- $(\Diamond A \supset \Box B) \supset \Box(A \supset B)$;

$$\frac{A \supset B \qquad A}{B}\,[mp] \qquad \frac{A}{\Box A}\,[nec]$$

In this paper, we consider the intuitionistic modal logics obtained from the combinations of the axioms T, B, 4 and 5 defined as follows:

(T) $(\Box A \supset A) \wedge (A \supset \Diamond A)$
(B) $(\Diamond\Box A \supset A) \wedge (A \supset \Box\Diamond A)$
(4) $(\Box A \supset \Box\Box A) \wedge (\Diamond\Diamond A \supset \Diamond A)$
(5) $(\Diamond\Box A \supset \Box A) \wedge (\Diamond A \supset \Box\Diamond A)$

For every $\mathsf{Th} \subseteq \{T, B, 4, 5\}$, we denote by IKTh the logic obtained by adding the axioms in Th to IK. For instance the logics $\mathsf{IK}T$, $\mathsf{IK}T4$ and $\mathsf{IK}T5$ are respectively the intuitionistic modal logics denoted in the literature by IT, $\mathsf{IS4}$ and $\mathsf{IS5}$.

Definition 1. *An* intuitionistic modal model *is a quadruple* $(W, \leqslant, \{D_w\}_{w\in W}, \{R_w\}_{w\in W}, \{V_w\}_{w\in W})$ *where*

- W *is a non-empty set of* Kripke worlds*;*
- $\leqslant$ *is a partial order relation on* W*;*
- *for any* $w \in W$*,* D_w *is a non-empty set of* modal worlds *such that if* $w \leqslant w'$ *then* $D_w \subseteq D_{w'}$*;*
- *for any* $w \in W$*,* R_w *is a binary relation on* D_w*, called* w-accessibility relation*, such that if* $w \leqslant w'$ *then* $R_w \subseteq R_{w'}$*;*
- *for any* $w \in W$*,* V_w *is a function from* D_w *to* 2^{Prop} *such that if* $w \leqslant w'$ *then* $V_w(p) \subseteq V_{w'}(p)$*.*

Let us mention that there are two kinds of worlds: the Kripke worlds (W) that correspond to the intuitionistic basis and the modal worlds (D_w) that capture the modal aspects.

We associate to each model $(W, \leqslant, \{D_w\}_{w\in W}, \{R_w\}_{w\in W}, \{V_w\}_{w\in W})$ a *forcing relation*, denoted $\vDash_{\mathcal{M}}$, that is inductively defined as follows:

- $w, d \vDash_{\mathcal{M}} p$ iff $p \in V_w(d)$;
- $w, d \vDash_{\mathcal{M}} \bot$ never;
- $w, d \vDash_{\mathcal{M}} A \wedge B$ iff $w, d \vDash_{\mathcal{M}} A$ and $w, d \vDash_{\mathcal{M}} B$;
- $w, d \vDash_{\mathcal{M}} A \vee B$ iff $w, d \vDash_{\mathcal{M}} A$ or $w, d \vDash_{\mathcal{M}} B$;
- $w, d \vDash_{\mathcal{M}} A \supset B$ iff for all $w' \geqslant w$, if $w', d \vDash_{\mathcal{M}} A$ then $w', d \vDash_{\mathcal{M}} B$;
- $w, d \vDash_{\mathcal{M}} \Box A$ iff for all $w' \geqslant w$ and for all $d' \in D_{w'}$, if $R_{w'}(d, d')$ then $w', d' \vDash_{\mathcal{M}} A$;
- $w, d \vDash_{\mathcal{M}} \Diamond A$ iff there exists $d' \in D_w$ such that $R_w(d, d')$ and $w, d' \vDash_{\mathcal{M}} A$.

A formula A is valid in a model $\mathcal{M} = (W, \leqslant, \{D_w\}_{w\in W}, \{R_w\}_{w\in W}, \{V_w\}_{w\in W})$ if and only if for all $w \in W$ and for all $d \in D_w$ $w, d \vDash_{\mathcal{M}} A$.

For $\mathsf{Th} \subseteq \{T, B, 4, 5\}$, the class of models defining the logics IKTh, denoted IC_{Th}, corresponds to the models in which the accessibility relations satisfy the properties associated to the axioms in Th:
(T) Reflexivity: $\forall w.R(w, w)$;
(B) Symmetry: $\forall w, w'.R(w, w') \supset R(w', w)$;
(4) Transitivity: $\forall w, w', w''.(R(w, w') \wedge R(w', w'')) \supset R(w, w'')$;
(5) Euclidness: $\forall w, w', w''.(R(w, w') \wedge R(w, w'')) \supset R(w', w'')$.

Theorem 1. *A formula* A *is valid in* IKTh *iff* A *is valid in every model in* IC_{Th}*.*

Proof. See [27]. ■

The forcing relation satisfies the property of Kripke monotonicity in the same way as in intuitionistic logic:

Proposition 1 (Monotonicity). *If* $w, d \vDash_{\mathcal{M}} A$ *and* $w \leqslant w'$ *then,* $w', d \vDash_{\mathcal{M}} A$ *holds.*

Proof. By structural induction on A. ■

Let us note that these logics do not satisfy the finite model property w.r.t. Kripke semantics [23, 27]. But some of them satisfy this property w.r.t. other semantics. The property has been proved for $\mathsf{IS5}$ [9] w.r.t. the algebraic semantics defined in [5]. Moreover, for the logics IK, $\mathsf{IK}B$ and $\mathsf{IK}TB$, the finite model property has been proved w.r.t. the bi-relational semantics [27].

We now define two useful notions, namely the size and the nesting degree of a formula. The *size* of a formula F, denoted $|F|$, is defined as follows: $|p| = |\bot| = 1$; $|A \otimes B| = |A| + |B| + 1$ where $\otimes \in \{\wedge, \vee, \supset\}$; $|\boxtimes A| = |A| + 1$ where $\boxtimes \in \{\Box, \Diamond\}$. $|F|$ is in fact the number of the subformula occurrences of F. The *nesting degree* of a formula F, denoted $nest(F)$, is defined as follows: $nest(p) = nest(\bot) = 0$; $nest(A \otimes B) = max(nest(A), nest(B))$ where $\otimes \in \{\wedge, \vee, \supset\}$ and max means the maximum; $nest(\boxtimes A) = 1 + nest(A)$ where $\boxtimes \in \{\Box, \Diamond\}$.

2.2 Sequent calculi for modal logics

In this paper we aim at defining label-free sequent calculi for all the intuitionistic modal logics obtained from the combinations of the axioms T, B, 4 and 5. In the literature, the sequent calculi for these logics are rare [2, 22, 27] and, as far as we know, the only approach that provide sequent calculi for all these modal logics is given in [27]. It explicitly integrates some semantic information, like the accessibility relation, into the systems by using labels and relations on labels. They allow us to define simple systems for a large number of modal logics, but these systems do not satisfy the subformula property. Moreover the interpretation of proofs is difficult, because of labels that are external to the logic language.

Label-free sequent calculi have been defined for the classical modal logics and they satisfy cut-elimination and subformula properties [3, 4]. They are based on a structure, named deep sequent, that can be seen as a generalization of the hypersequent structure for the modal logic $\mathsf{S5}$ [1]. However, deep sequent and hypersequent structures defined for classical modal logics are not appropriate to deal with the intuitionistic modal logics. This is mainly due to the disjunction property satisfied by the intuitionistic modal logics that says if $A \vee B$ is a theorem, then A is a theorem or B is theorem. Let us note that this property is not satisfied in the classical case, for instance, $A \vee \neg A$ being a theorem without neither A nor $\neg A$ being theorems. Deep sequent and hypersequent structures are based on disjunction in the sense that they are multi-contextual structures where the contexts are separated by disjunctions, and thus they do not really extend the standard sequent structure in the intuitionistic case because of the disjunction property.

Thus we propose a multi-contextual sequent structure introduced in [14] for the definition of natural deduction systems. The present paper provides label-free sequent calculi for intuitionistic modal logics on the basis of this structure. It can be seen as a continuation of our previous works in order to define proof systems adapted to proof-search in these logics.

3 The Tree-sequent structure

In this section we present the multi-contextual structure, called Tree-sequent (or T-sequent), that we first introduced in [14]. It is different from the one of deep (or nested) sequent [3, 4] used for classical modal logics, even if in a sense it can be seen as its mono-conclusion version.

A *T-context* is a syntactic structure defined by induction as follows:

- if Γ is a multiset of formulae, then Γ is a T-context;
- if Γ is a T-context, then $\langle\Gamma\rangle$ is a T-context;
- if Γ and Γ' are T-contexts, then Γ, Γ' is a T-context.

In other words, a T-context is a structure of the form $A_1, \ldots, A_k, \langle\Gamma_1\rangle, \ldots, \langle\Gamma_l\rangle$ where $A_1, \ldots, A_k$ is a (possibly empty) multiset of formulae and $\Gamma_1, \ldots, \Gamma_l$ is a (possibly empty) multiset of T-contexts.
A *marked formula* is an expression of the form $A^{\vdash}$ where A is a formula.

Definition 2 (T-sequent). *A* T-sequent *is inductively defined as follows:*

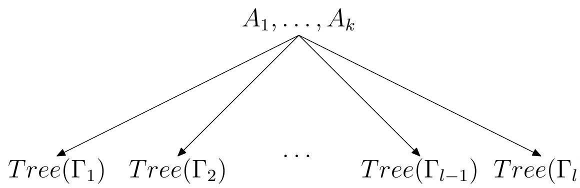
- *If Γ is a T-context and $A^{\vdash}$ is a marked formula then $\Gamma, A^{\vdash}$ is a T-sequent.*
- *If $\mathcal{S}$ is a T-sequent and Γ is a T-context then $\Gamma, \langle\mathcal{S}\rangle$ is a T-sequent.*

A T-sequent has the same form as a T-context, indeed, it can be seen as a T-context with in addition only one occurrence of a marked formula, that is called the *conclusion*.

One can associate a formula to each T-context with $\mathcal{F}$ defined by : $\mathcal{F}(\emptyset) = \top$ and $\mathcal{F}(A_1, \ldots, A_k, \langle\Gamma_1\rangle, \ldots, \langle\Gamma_l\rangle) = A_1 \wedge \ldots \wedge A_k \wedge \Diamond(\mathcal{F}(\Gamma_1)) \wedge \ldots \wedge \Diamond(\mathcal{F}(\Gamma_l))$.
It is extended to the T-sequents as follows:
$\mathcal{F}(\Gamma, A^{\vdash}) = \mathcal{F}(\Gamma) \supset A$. $\mathcal{F}(\Gamma, \langle\mathcal{S}\rangle) = \mathcal{F}(\Gamma) \supset \Box(\mathcal{F}(\mathcal{S}))$ with Γ being a T-context and $\mathcal{S}$ a T-sequent.

For instance we have $\mathcal{F}(\Box(A\supset B), \Diamond A, \langle A, B^{\vdash}\rangle) = (\Box(A\supset B)\wedge(\Diamond A))\supset\Box(A\supset B)$.

The validity of a T-sequent $\mathcal{S}$ in a modal logic L is defined as the validity of its corresponding formula $\mathcal{F}(\mathcal{S})$ in L. The T-contexts and T-sequents can also be presented graphically. For instance, $A_1, \ldots, A_k, \langle\Gamma_1\rangle, \ldots, \langle\Gamma_l\rangle$ is represented by the following figure:



where $Tree(\Gamma_i)$ is the tree corresponding to Γ_i.

Let us note that we do not distinguish the T-sequents and T-contexts and their associated trees. When we mention the *root*, a *leaf*, the *depth* or a *subtree* of a T-sequent or a T-context, we refer to its associated tree.

A *nT-context*, with $n \geqslant 0$, is a T-context or a T-sequent with exactely n occurrences of the symbol $\{\}$, called a T-hole.

It is denoted $\Gamma\overbrace{\{\}\cdots\{\}}^{n\times}$ by considering that a bijection maps an occurrence of $\{\}$ in the nT-context to each occurrence of the symbol $\{\}$ following this notation. The structure $\Gamma\{\Delta_1\}\cdots\{\Delta_n\}$ is obtained by the substitution of the T-hole associated to the *ith* occurrence of $\{\}$ in $\Gamma\{\}\cdots\{\}$ by Δ_i, for every $i \in \{1, \ldots, n\}$. For instance, any T-sequent has the form $\Gamma\{C^{\vdash}\}$ where $\Gamma\{\}$ is a $1T$-context. From now we denote $\Gamma\{\emptyset\}$ the T-context of $\Gamma\{C^{\vdash}\}$.
In general the T-holes are substituted by T-contexts, T-sequents or nT-contexts. As an example the T-sequent $\Box(A \supset B), \Diamond A, \langle A, B^{\vdash}\rangle$ corresponds to $\Gamma\{B^{\vdash}\}$ where $\Gamma\{\} = \Box(A \supset B), \Diamond A, \langle A, \{\}\rangle$.

A T-sequent can be seen as a multi-contextual structure because the truth value of a T-sequent may change w.r.t. the position (context) of its conclusion in the tree associated to its T-context. Moreover it can be seen as a kind of mono-conclusion version of a deep (or nested) sequent [3, 4], but it is different in two main ways.

Let us remind that a deep sequent is a structure $A_1, \ldots, A_k, [\Gamma_1], \ldots, [\Gamma_n]$ where $\{A_1, \ldots, A_k\}$ is a multiset of formulae and Γ_i are contexts. In a deep sequent there is no distinction between formulae but in a T-sequent we distinguish one formula, the marked one, as a conclusion knowing that the others are considered as hypotheses. Moreover the formula associated to a deep sequent is $A_1 \vee \cdots \vee A_k \vee \Box(\mathcal{F}(\Gamma_1)) \vee \ldots \vee \Box(\mathcal{F}(\Gamma_n))$ and it only deals with the modal connective $\Box$ while the formula associated to a T-sequent considers both modal connectives $\Diamond$ and $\Box$. These key points emphasize why the T-sequent structure is well adapted for studying structural proof theory in intuitionistic modal logics.

We end this section by giving definitions useful in the rest of the paper. The depth of a $1T$-context $\Gamma\{\}$, denoted $depth(\Gamma\{\})$, is defined as follows: $depth(\Gamma, \{\}) = 0$; $depth(\Gamma, \langle\Delta\{\}\rangle) = 1 + depth(\Delta\{\})$. Let $\mathcal{S}$ be a T-sequent, $sp(\mathcal{S})$ is a unary relation that is true if and only if the depth of the tree cor-

$$\frac{}{\Gamma\{A, A^{\vdash}\}}\ [id] \qquad \frac{\Gamma\{\bot^{\vdash}\}\{\emptyset\}}{\Gamma\{\emptyset\}\{A^{\vdash}\}}\ [\bot_E]$$

$$\frac{\Gamma\{A^{\vdash}\} \qquad \Gamma\{B^{\vdash}\}}{\Gamma\{A \wedge B^{\vdash}\}}\ [\wedge_I] \qquad \frac{\Gamma\{A \wedge B^{\vdash}\}}{\Gamma\{A^{\vdash}\}}\ [\wedge_E^1] \qquad \frac{\Gamma\{A \wedge B^{\vdash}\}}{\Gamma\{B^{\vdash}\}}\ [\wedge_E^2]$$

$$\frac{\Gamma\{A^{\vdash}\}}{\Gamma\{A \vee B^{\vdash}\}}\ [\vee_I^1] \quad \frac{\Gamma\{B^{\vdash}\}}{\Gamma\{A \vee B^{\vdash}\}}\ [\vee_I^2] \quad \frac{\Gamma\{A \vee B^{\vdash}\}\{\emptyset\} \quad \Gamma\{A\}\{C^{\vdash}\} \quad \Gamma\{B\}\{C^{\vdash}\}}{\Gamma\{\emptyset\}\{C^{\vdash}\}}\ [\vee_E]$$

$$\frac{\Gamma\{A, B^{\vdash}\}}{\Gamma\{A \supset B^{\vdash}\}}\ [\supset_I] \qquad \frac{\Gamma\{A \supset B^{\vdash}\} \qquad \Gamma\{A^{\vdash}\}}{\Gamma\{B^{\vdash}\}}\ [\supset_E]$$

$$\frac{\Gamma\{\langle \Delta, A^{\vdash}\rangle\}}{\Gamma\{\langle \Delta\rangle, \Diamond A^{\vdash}\}}\ [\Diamond_I] \qquad \frac{\Gamma\{\Diamond A^{\vdash}\}\{\emptyset\} \qquad \Gamma\{\langle A\rangle\}\{C^{\vdash}\}}{\Gamma\{\emptyset\}\{C^{\vdash}\}}\ [\Diamond_E]$$

$$\frac{\Gamma\{\langle A^{\vdash}\rangle\}}{\Gamma\{\Box A^{\vdash}\}}\ [\Box_I] \qquad \frac{\Gamma\{\langle \Delta\rangle, \Box A^{\vdash}\}}{\Gamma\{\langle \Delta, A^{\vdash}\rangle\}}\ [\Box_E]$$

Fig. 1. A Natural Deduction System ND_{IK}

responding to $\mathcal{S}$ is greater than 0. Moreover we define $nest(\mathcal{S})$ by $nest(\mathcal{S}) = max\{nest(A) \mid A \in \mathcal{S}\}$ where max means the maximum and $nest(A)$ the nesting degree of A previously defined. Moreover $d(\mathcal{S})$ represents the depth of $\mathcal{S}$.

4 A natural deduction system ND_{IKTh}

In this section, we present a natural deduction system for IK, called ND_{IK}, that is based on the T-sequent structure. It has been defined in [14] and has the normalization property. The natural deduction system ND_{IK} is given in Figure 1.

We associate to each logic IKTh, with $\mathsf{Th} \subseteq \{T, B, 4, 5\}$, the natural deduction system ND_{IKTh} that is obtained by using the rules described in Figure 2 as follows:

- if IKTh is IS5 then ND_{IKTh} is obtained from ND_{IK} by replacing the rules $[\Box_E]$ and $[\Diamond_I]$ by the rules $[\Box_E^{\mathsf{IS5}}]$ and $[\Diamond_I^{\mathsf{IS5}}]$;
- if IKTh is IB4 then ND_{IKTh} is obtained from ND_{IK} by replacing the rules $[\Box_E]$ and $[\Diamond_I]$ by the rules $[\Box_E^{\mathsf{IB4}}]$ and $[\Diamond_I^{\mathsf{IB4}}]$;
- otherwise ND_{IKTh} is obtained by adding to ND_{IK} the rules $[\Box_E^x]$ and $[\Diamond_I^x]$ for any $x \in \mathsf{Th}$.

The rules of ND_{IKTh} are all of the following form:

$$\frac{\Gamma\{\Delta_1^1\}\cdots\{\Delta_k^1\} \quad \cdots \quad \Gamma\{\Delta_1^l\}\cdots\{\Delta_k^l\}}{\Gamma\{\Delta_1\}\cdots\{\Delta_k\}}\ [R]$$

$$\frac{\Gamma\{\Box A^{\vdash}\}}{\Gamma\{A^{\vdash}\}}\ [\Box_E^T] \qquad \frac{\Gamma\{A^{\vdash}\}}{\Gamma\{\Diamond A^{\vdash}\}}\ [\Diamond_I^T]$$

$$\frac{\Gamma\{\langle\Delta, \Box A^{\vdash}\rangle\}}{\Gamma\{\langle\Delta\rangle, A^{\vdash}\}}\ [\Box_E^B] \qquad \frac{\Gamma\{\langle\Delta\rangle, A^{\vdash}\}}{\Gamma\{\langle\Delta, \Diamond A^{\vdash}\rangle\}}\ [\Diamond_I^B]$$

$$\frac{\Gamma\{\Delta\{\emptyset\}, \Box A^{\vdash}\}}{\Gamma\{\Delta\{A^{\vdash}\}\}}\ [\Box_E^4](*) \qquad \frac{\Gamma\{\Delta\{A^{\vdash}\}\}}{\Gamma\{\Delta\{\emptyset\}, \Diamond A^{\vdash}\}}\ [\Diamond_I^4](*)$$

$$\frac{\Gamma\{\Box A^{\vdash}\}\{\emptyset\}}{\Gamma\{\emptyset\}\{A^{\vdash}\}}\ [\Box_E^5](**) \qquad \frac{\Gamma\{\emptyset\}\{A^{\vdash}\}}{\Gamma\{\Diamond A^{\vdash}\}\{\emptyset\}}\ [\Diamond_I^5](**)$$

$$\frac{\Gamma\{\Box A^{\vdash}\}\{\emptyset\}}{\Gamma\{\emptyset\}\{A^{\vdash}\}}\ [\Box_E^{\mathsf{IB4}}](***) \qquad \frac{\Gamma\{\emptyset\}\{A^{\vdash}\}}{\Gamma\{\Diamond A^{\vdash}\}\{\emptyset\}}\ [\Diamond_I^{\mathsf{IB4}}](***)$$

$$\frac{\Gamma\{\Box A^{\vdash}\}\{\emptyset\}}{\Gamma\{\emptyset\}\{A^{\vdash}\}}\ [\Box_E^{\mathsf{IS5}}] \qquad \frac{\Gamma\{\emptyset\}\{A^{\vdash}\}}{\Gamma\{\Diamond A^{\vdash}\}\{\emptyset\}}\ [\Diamond_I^{\mathsf{IS5}}]$$

(*) $depth(\Delta\{\}) > 1$;
(**) $depth(\Gamma\{\}\{\emptyset\}) > 0$ *and* $depth(\Gamma\{\emptyset\}\{\}) > 0$;
(***) $sp(\Gamma\{\Box A^{\vdash}\}\{\emptyset\})$.

Fig. 2. Modal rules for ND_{IKTh}

It means that each premise is obtained by the transformation of some subtrees of the conclusion. In an elimination rule we call *major premise* the premise that contains the eliminated connective and the other premises are called *minor premises*. A *detour* in a natural deduction proof corresponds to an application of a rule that introduces a logical operator followed by an application of a rule that eliminates it. A proof is in *normal form* if it is without detours. The main goal of the normalization property is the elimination of all detours in a proof.

Definition 3 (Segment). *A segment of length n in a proof $\mathcal{D}$ in ND_{IKTh} is a sequence $\Gamma_1\{A^{\vdash}\}, \ldots, \Gamma_n\{A^{\vdash}\}$ of consecutive occurrences of T-sequents in $\mathcal{D}$ such that:*

- *for $n > 1$ and $i < n$, $\Gamma_i\{A^{\vdash}\}$ is a minor premise of an application of $[\vee_E]$ or $[\Diamond_E]$ in $\mathcal{D}$ with the conclusion $\Gamma_{i+1}\{A^{\vdash}\}$;*
- *$\Gamma_1\{A^{\vdash}\}$ is not the conclusion of an application of $[\vee_E]$ or of $[\Diamond_E]$;*
- *$\Gamma_n\{A^{\vdash}\}$ is not a minor premise of an application of $[\vee_E]$ or $[\Diamond_E]$.*

T-sequents of a segment have the same conclusion, called the conclusion of the segment. A segment is a premise (resp. the conclusion) of a rule application if its last element (resp. first element) is a premise (resp. the conclusion) of this application.

Definition 4 (Normal Form). *A segment* $\Gamma_1\{A^{\vdash}\}, \ldots, \Gamma_n\{A^{\vdash}\}$ *is a* cut *if* $\Gamma_n\{A^{\vdash}\}$ *is the main premise of the application of an elimination rule, and either* $n > 1$ *or* $n = 1$ *and* $\Gamma_1\{A^{\vdash}\}$ *is the conclusion of an introduction rule or of the rule* $[\bot_E]$*. A proof is in* normal form *if it does not contain such a cut.*

Theorem 2 ([14]). *Any proof in* ND_{IKTh} *can be reduced to a proof in normal form.*

Let us remind that a *main branch* of a derivation $\mathcal{D}$ in ND_{IKTh} is a branch which begins with a leaf, ends with the conclusion of $\mathcal{D}$ and passes only through premises of introduction rules and major premises of elimination rules. Note that if a proof in normal form has as last rule application an elimination rule, then it contains only one main branch. Indeed, this is a consequence of the fact that such a proof contains only applications of elimination rules. In fact, any proof in normal form does not contain an application of $[\wedge_I]$ and contains only one main branch, because $[\wedge_I]$ is the unique introduction rule having two premises.

5 Sequent calculi for intuitionistic modal logics

In this section we propose a quasi-modular sequent calculus and then the resulting calculi for the intuitionistic modal logics obtained by the combinations of the axioms T, B, 4 and 5. The modularity is based on the association of specific rules to the axioms.

5.1 The G_{IKTh} sequent calculus

In this subsection we first present a Tree-sequent calculus, called G_{IK}, for the IK logic. It is given in Figure 3. In the case of a standard sequent calculus, the set of rules associated to each connective are composed from *left* and *right* rules. The left rules act on the left of the symbol $\vdash$, while the right ones act on the right of this symbol. In the case of our sequent calculi, we consider the rules acting on the T-context as left rules and the ones acting on the marked formula as right rules. In every sequent calculus rule, each premise is obtained by the transformation of some subtrees of the conclusion.

After introducing our sequent calculus for IK, we present a quasi-modular sequent calculus G_{IKTh} from this sequent calculus for G_{IK}. For that we associate to each logic IKTh, with $\mathsf{Th} \subseteq \{T, B, 4, 5\}$, the sequent calculus G_{IKTh} that is obtained by using the rules described in Figure 4 as follows:

- if IKTh is IS5 then G_{IKTh} is obtained from G_{IK} by replacing the rules $[\Box_L]$ and $[\Diamond_R]$ by the rules $[\Box_L^{\mathsf{IS5}}]$ and $[\Diamond_R^{\mathsf{IS5}}]$;
- if IKTh is IB4 then G_{IKTh} is obtained from G_{IK} by replacing the rules $[\Box_L]$ and $[\Diamond_R]$ by the rules $[\Box_L^{\mathsf{IB4}}]$ and $[\Diamond_R^{\mathsf{IB4}}]$;
- otherwise G_{IKTh} is obtained by adding to G_{IK} the rules $[\Box_L^x]$ and $[\Diamond_R^x]$ for all $x \in \mathsf{Th}$.

$$\frac{}{\Gamma\{A, A^{\vdash}\}}\ [id] \qquad \frac{}{\Gamma\{\bot\}\{C^{\vdash}\}}\ [\bot_L]$$

$$\frac{\Gamma\{A_1 \wedge A_2, A_1\}\{C^{\vdash}\}}{\Gamma\{A_1 \wedge A_2\}\{C^{\vdash}\}}\ [\wedge_L^1] \qquad \frac{\Gamma\{A_1 \wedge A_2, A_2\}\{C^{\vdash}\}}{\Gamma\{A_1 \wedge A_2\}\{C^{\vdash}\}}\ [\wedge_L^2] \qquad \frac{\Gamma\{A^{\vdash}\} \qquad \Gamma\{B^{\vdash}\}}{\Gamma\{A \wedge B^{\vdash}\}}\ [\wedge_R]$$

$$\frac{\Gamma\{A \vee B, A\}\{C^{\vdash}\} \qquad \Gamma\{A \vee B, B\}\{C^{\vdash}\}}{\Gamma\{A \vee B\}\{C^{\vdash}\}}\ [\vee_L] \qquad \frac{\Gamma\{A_i^{\vdash}\}}{\Gamma\{A_1 \vee A_2^{\vdash}\}}\ [\vee_R^i]$$

$$\frac{\Gamma\{A \supset B, A^{\vdash}\}\{\emptyset\} \qquad \Gamma\{A \supset B, B\}\{C^{\vdash}\}}{\Gamma\{A \supset B\}\{C^{\vdash}\}}\ [\supset_L] \qquad \frac{\Gamma\{A, B^{\vdash}\}}{\Gamma\{A \supset B^{\vdash}\}}\ [\supset_R]$$

$$\frac{\Gamma\{\langle A\rangle, \Diamond A\}\{C^{\vdash}\}}{\Gamma\{\Diamond A\}\{C^{\vdash}\}}\ [\Diamond_L] \qquad \frac{\Gamma\{\langle \Delta, A^{\vdash}\rangle\}}{\Gamma\{\langle \Delta\rangle, \Diamond A^{\vdash}\}}\ [\Diamond_R]$$

$$\frac{\Gamma\{\langle \Delta, A\rangle, \Box A\}}{\Gamma\{\langle \Delta\rangle, \Box A\}}\ [\Box_L] \qquad \frac{\Gamma\{\langle A^{\vdash}\rangle\}}{\Gamma\{\Box A^{\vdash}\}}\ [\Box_R]$$

$$\frac{\Gamma\{A^{\vdash}\}\{\emptyset\} \qquad \Gamma\{A\}\{C^{\vdash}\}}{\Gamma\{\emptyset\}\{C^{\vdash}\}}\ [Cut]$$

Fig. 3. The Tree-sequent calculus G_{IK}

We denote by G^{-}_{IKTh} the calculus G_{IKTh} without the $[Cut]$ rule (also called cut rule).

A system is said modular if we have a system for IK such that for any subset Th of $\{T, B, 4, 5\}$, the addition of rules associated to axioms in Th leads to a system for the logic IKTh. For instance, in the case of classical modal logics based on these axioms, a modular calculus, based on deep sequents, has been recently defined [3]. Our calculus is said quasi-modular because the logics IB4 and IS5 are separately studied.

Let us note that the rules $[\Box_L^4]$, $[\Diamond_R^4]$, $[\Box_L^5]$, $[\Diamond_R^5]$, $[\Box_L^{\mathsf{IB4}}]$ and $[\Diamond_R^{\mathsf{IB4}}]$ can be applied only if some conditions are satisfied. Intuitively, these conditions capture the properties of the accessibility relations in Kripke semantics.
We can apply the rule $[\Box_L^4]$ if it satisfies the condition $depth(\Delta\{\}) > 1$. This condition allows us to only consider the transitivity property. Indeed, if we apply the rule $[\Box_L^4]$ with the condition $depth(\Delta\{\}) = 0$, then it means that the accessibility relation is reflexive. Moreover, an application of $[\Box_L^4]$ with $depth(\Delta\{\}) = 1$ corresponds to an application of $[\Box_L]$.

$$\frac{\Gamma\{\Box A, A\}}{\Gamma\{\Box A\}}\ [\Box_L^T] \qquad \frac{\Gamma\{A^{\vdash}\}}{\Gamma\{\Diamond A^{\vdash}\}}\ [\Diamond_R^T]$$

$$\frac{\Gamma\{\langle \Delta, \Box A\rangle, A\}}{\Gamma\{\langle \Delta, \Box A\rangle\}}\ [\Box_L^B] \qquad \frac{\Gamma\{\langle \Delta\rangle, A^{\vdash}\}}{\Gamma\{\langle \Delta, \Diamond A^{\vdash}\rangle\}}\ [\Diamond_R^B]$$

$$\frac{\Gamma\{\Delta\{A\}, \Box A\}}{\Gamma\{\Delta\{\emptyset\}, \Box A\}}\ [\Box_L^4](\dagger) \qquad \frac{\Gamma\{\Delta\{A^{\vdash}\}\}}{\Gamma\{\Delta\{\emptyset\}, \Diamond A^{\vdash}\}}\ [\Diamond_R^4](\dagger)$$

$$\frac{\Gamma\{\Box A\}\{A\}}{\Gamma\{\Box A\}\{\emptyset\}}\ [\Box_L^5](\dagger\dagger) \qquad \frac{\Gamma\{\emptyset\}\{A^{\vdash}\}}{\Gamma\{\Diamond A^{\vdash}\}\{\emptyset\}}\ [\Diamond_R^5](\dagger\dagger)$$

$$\frac{\Gamma\{\Box A\}\{A\}}{\Gamma\{\Box A\}\{\emptyset\}}\ [\Box_L^{\mathsf{IB4}}](\dagger\dagger\dagger) \qquad \frac{\Gamma\{\emptyset\}\{A^{\vdash}\}}{\Gamma\{\Diamond A^{\vdash}\}\{\emptyset\}}\ [\Diamond_R^{\mathsf{IB4}}](\dagger\dagger\dagger)$$

$$\frac{\Gamma\{\Box A\}\{A\}}{\Gamma\{\Box A\}\{\emptyset\}}\ [\Box_L^{\mathsf{IS5}}] \qquad \frac{\Gamma\{\emptyset\}\{A^{\vdash}\}}{\Gamma\{\Diamond A^{\vdash}\}\{\emptyset\}}\ [\Diamond_R^{\mathsf{IS5}}]$$

($\dagger$) $depth(\Delta\{\}) > 1$;
($\dagger\dagger$) $depth(\Gamma\{\}\{\emptyset\}) > 0$ *and* $depth(\Gamma\{\emptyset\}\{\}) > 0$;
($\dagger\dagger\dagger$) $sp(\Gamma\{\emptyset\}\{\emptyset\})$.

Fig. 4. Modal rules for G_{IKTh}

5.2 Properties of the G_{IKTh} calculus

The G_{IKTh} sequent calculus is proved sound and complete by considering validity in IKTh with the natural deduction system ND_{IKTh} defined in [14]. In other words, we prove that a T-sequent has a proof in G_{IKTh} if and only if it has a proof in ND_{IKTh}.

Theorem 3 (Soundness). *If a T-sequent has a proof in G_{IKTh}, then it has a proof in ND_{IKTh}.*

Proof. Let $\mathcal{S}$ be a T-sequent and $\mathcal{D}$ be a proof of $\mathcal{S}$ in G_{IKTh}. We show by induction on the structure of $\mathcal{D}$ that $\mathcal{S}$ has a proof in ND_{IKTh}.
The case where $\mathcal{D}$ has a right rule as last rule application is trivial, because the right rules of G_{IKTh} and the introduction rules of ND_{IKTh} are the same.
Since $[Cut]$ is an admissible rule in ND_{IKTh} (see [14]), the case where $\mathcal{D}$ has a $[Cut]$ as last rule application is also trivial.

Regarding the cases where $\mathcal{D}$ has a left rule as last rule application we only develop the $[\wedge_L]$ and $[\Diamond_L]$ cases, the others being similar. Let us denote $[H.I]$ any application of the induction hypothesis.

– $[\wedge_L^i]$:

$$\frac{\Gamma\{A_1 \wedge A_2, A_i\}\{C^{\vdash}\}}{\Gamma\{A_1 \wedge A_2\}\{C^{\vdash}\}}\ [\wedge_L]$$

A proof of $\Gamma\{A_1 \wedge A_2\}\{C^{\vdash}\}$ in ND_{IKTh} is obtained as follows:

$$\dfrac{\dfrac{\overline{\Gamma\{A_1 \wedge A_2, A_1 \wedge A_2^{\vdash}\}\{\emptyset\}}\ [id]}{\Gamma\{A_1 \wedge A_2, A_i^{\vdash}\}\{\emptyset\}}\ [\wedge_E^i] \quad \overline{\Gamma\{A_1 \wedge A_2, A_i\}\{C^{\vdash}\}}\ [H.I]}{\Gamma\{A_1 \wedge A_2\}\{C^{\vdash}\}}\ [Cut]$$

– $[\Diamond_L]$:

$$\frac{\Gamma\{\langle A\rangle, \Diamond A\}\{C^{\vdash}\}}{\Gamma\{\Diamond A\}\{C^{\vdash}\}}\ [\Diamond_L]$$

A proof of $\Gamma\{\Diamond A\}\{C^{\vdash}\}$ in ND_{IKTh} is obtained as follows:

$$\dfrac{\overline{\Gamma\{\Diamond A, \Diamond A^{\vdash}\}\{\emptyset\}}\ [id] \quad \overline{\Gamma\{\langle A\rangle, \Diamond A\}\{C^{\vdash}\}}\ [H.I]}{\Gamma\{\Diamond A\}\{C^{\vdash}\}}\ [\Diamond_E]$$

■

Having proved soundness we now consider completeness. In order to prove it we consider the following proposition that expresses that the rule $[\bot_E]$ of ND_{IKTh} is admissible in G^{-}_{IKTh}. It is used in our proof of completeness.

Proposition 2 (Admissibility of $[\bot_E]$). *If $\Gamma\{\bot^{\vdash}\}\{\emptyset\}$ has a proof in G^{-}_{IKTh}, then $\Gamma\{\emptyset\}\{A^{\vdash}\}$ has also a proof in G^{-}_{IKTh}.*

Proof. Let $\mathcal{D}$ be a proof of $\Gamma\{\bot^{\vdash}\}\{\emptyset\}$. The proof is by structural induction on $\mathcal{D}$. If $\Gamma\{\bot^{\vdash}\}\{\emptyset\}$ is an axiom, then $\Gamma\{\emptyset\}\{A^{\vdash}\}$ is also an axiom (an instance of $[\bot_L]$). Otherwise, a proof of $\Gamma\{\emptyset\}\{A^{\vdash}\}$ is obtained by an application of induction hypothesis on the premise having $\bot$ as marked formula in the last rule application in $\mathcal{D}$. For example, if this rule application corresponds to an application of $[\supset_L]$:

$$\frac{\Gamma\{B \supset C, B^{\vdash}\}\{\emptyset\}\{\emptyset\} \qquad \Gamma\{B \supset C, C\}\{\bot^{\vdash}\}\{\emptyset\}}{\Gamma\{B \supset C\}\{\bot^{\vdash}\}\{\emptyset\}}\ [\supset_L]$$

then, by induction hypothesis, we know that $\Gamma\{B \supset C, C\}\{\emptyset\}\{A^{\vdash}\}$ has a proof in G^{-}_{IKTh}. Finally by using $[\supset_L]$ with $\Gamma\{B \supset C, B^{\vdash}\}\{\emptyset\}\{\emptyset\}$ and $\Gamma\{B \supset C, C\}\{\emptyset\}\{A^{\vdash}\}$ as premises we obtain a proof of $\Gamma\{B \supset C\}\{\emptyset\}\{A^{\vdash}\}$. ■

Theorem 4 (Completeness). *If a T-sequent has a proof in ND_{IKTh}, then it has a proof in G^{-}_{IKTh}.*

Proof. Let $\mathcal{S}$ be a T-sequent and $\mathcal{D}$ a proof of $\mathcal{S}$ in ND_{IKTh} in normal form. This proof is by induction on the number of rule applications n in $\mathcal{D}$. We show that $\mathcal{S}$ has a proof in G^{-}_{IKTh} (without using the rule $[Cut]$).
- If $n = 1$ then $\mathcal{S}$ is an instance of $[id]$ in ND_{IKTh} and thus it is an instance of $[id]$ in G^{-}_{IKTh}.
- The case where $\mathcal{D}$ has an introduction rule as last rule application is trivial, because the right rules of G^{-}_{IKTh} and the introduction rules of ND_{IKTh} are the same.
- The case where $\mathcal{D}$ has the rule $[\bot_E]$ as last rule application is proved by using Proposition 2.
- Let us consider the cases where $\mathcal{D}$ has an elimination rule as last rule application . Let α be the main branch of $\mathcal{D}$ (see Section 4). Since α does not contain a minor premise, the T-context of the T-sequent $\mathcal{S}_0$ in the beginning of this branch and the one of $\mathcal{S}$ are the same. Indeed, the T-context of $\mathcal{S}_0$ is not modified by any rule application in the main branch. Therefore, there exist a $2T$-context $\Gamma\{\}\{\}$ and two formulae A and C such that $\mathcal{S}_0 = \Gamma\{A, A^{\vdash}\}\{\emptyset\}$ and $\mathcal{S} = \Gamma\{A\}\{C^{\vdash}\}$.

In the following, we consider the different cases for A. Let us note that A cannot be an atomic formula, because if it is then $\mathcal{S}_0$ is a premise of an introduction rule or $[\bot_E]$ and then $\mathcal{D}$ is not in normal form. We only consider the cases $A \equiv A_1 \supset A_2$, $A \equiv \Diamond A'$ and $A \equiv \Box A'$.

- $A \equiv A_1 \supset A_2$**:**
$\mathcal{D}$ is of the following normal form:

$$\dfrac{\dfrac{}{\Gamma\{A_1 \supset A_2, A_1 \supset A_2^{\vdash}\}\{\emptyset\}}[id] \qquad \begin{array}{c}\mathcal{D}_1\\ \Gamma\{A_1 \supset A_2, A_1^{\vdash}\}\{\emptyset\}\end{array}}{\begin{array}{c}\Gamma\{A_1 \supset A_2, A_2^{\vdash}\}\{\emptyset\}\\ \mathcal{D}_2\\ \Gamma\{A_1 \supset A_2\}\{C^{\vdash}\}\end{array}}[\supset_E]$$

where $\mathcal{D}_2$ corresponds to the sub-derivation of $\mathcal{D}$ obtained by removing the T-sequents before the application of $[\supset_E]$. Clearly, by adding A_2 in the node where $A_2^{\vdash}$ appears in the T-sequents in $\mathcal{D}_2$, we obtain a proof of $\Gamma\{A_1 \supset A_2, A_2\}\{C^{\vdash}\}$. The number of rule applications in $\mathcal{D}_1$ and in $\mathcal{D}_2$ are smaller than this in $\mathcal{D}$. Thus by induction hypothesis, $\Gamma\{A_1 \supset A_2, A_1^{\vdash}\}\{\emptyset\}$ and $\Gamma\{A_1 \supset A_2, A_2\}\{C^{\vdash}\}$ have proofs in G^{-}_{IKTh}. By taking the rule $[\supset_L]$ with $\Gamma\{A_1 \supset A_2, A_1^{\vdash}\}\{\emptyset\}$ and $\Gamma\{A_1 \supset A_2, A_2\}\{C^{\vdash}\}$ as premises, we obtain a proof of $\Gamma\{A_1 \supset A_2\}\{C^{\vdash}\}$ in G^{-}_{IKTh}.

- $A \equiv \Diamond A'$**:**
$\mathcal{D}$ is of the following normal form:

$$\dfrac{\dfrac{}{\Gamma\{\Diamond A', \Diamond A'^{\vdash}\}\{\emptyset\}}[id] \qquad \begin{array}{c}\mathcal{D}_1\\ \Gamma\{\Diamond A', \langle A'\rangle\}\{C^{\vdash}\}\end{array}}{\Gamma\{\Diamond A'\}\{C^{\vdash}\}}[\Diamond_E]$$

By induction hypothesis, $\Gamma\{\Diamond A', \langle A'\rangle\}\{C^{\vdash}\}$ has a proof in G^{-}_{IKTh}. By taking $[\Diamond_L]$ with $\Gamma\{\Diamond A', \langle A'\rangle\}\{C^{\vdash}\}$ as a premise, we obtain a proof of $\Gamma\{\Diamond A'\}\{C^{\vdash}\}$ in G^{-}_{IKTh}.

- $A \equiv \Box A'$:
If $\mathcal{D}$ is of the following normal form ($[\Box_E]$):

$$\dfrac{\dfrac{}{\Gamma'\{\langle\Delta\rangle, \Box A', \Box A'^{\vdash}\}}\,[id]}{\dfrac{\Gamma'\{\langle\Delta, A'^{\vdash}\rangle, \Box A'\}}{\begin{array}{c}\mathcal{D}'\\ \Gamma'\{\langle\Delta\rangle, \Box A'\}\end{array}}}\,[\Box_E]$$

where $\mathcal{D}'$ corresponds to the sub-derivation of $\mathcal{D}$ obtained by removing the T-sequents before the application of $[\Box_E]$ and $\Gamma'\{\langle\Delta\rangle, \Box A'\} = \Gamma\{\Box A'\}\{C^{\vdash}\}$. By adding A' in the node where $A'^{\vdash}$ appears in the T-sequents in $\mathcal{D}'$, we obtain a proof of $\Gamma'\{\langle\Delta, A'\rangle, \Box A'\}$. Since the number of rule applications in $\mathcal{D}'$ is smaller than this in $\mathcal{D}$, we deduce by induction hypothesis that $\Gamma'\{\langle\Delta, A'\rangle, \Box A'\}$ has a proof in G^{-}_{IKTh}. Using $[\Box_L]$ with $\Gamma'\{\langle\Delta, A'\rangle, \Box A'\}$ as a premise, we have a proof of $\Gamma'\{\langle\Delta\rangle, \Box A'\}$ in G^{-}_{IKTh}.

If $\mathcal{D}$ is of the following normal form ($[\Box^T_E]$):

$$\dfrac{\dfrac{}{\Gamma'\{\Box A', \Box A'^{\vdash}\}}\,[id]}{\dfrac{\Gamma'\{A'^{\vdash}, \Box A'\}}{\begin{array}{c}\mathcal{D}'\\ \Gamma'\{\Box A'\}\end{array}}}\,[\Box^T_E]$$

$\Gamma'\{\Box A'\} = \Gamma\{\Box A'\}\{C^{\vdash}\}$. By induction hypothesis, $\Gamma'\{A', \Box A'\}$ has a proof in G^{-}_{IKTh}. Using $[\Box^T_L]$ with $\Gamma'\{A', \Box A'\}$ as a premise we obtain a proof of $\Gamma'\{\Box A'\}$ in G^{-}_{IKTh}.

The cases of $[\Box^B_E]$, $[\Box^4_E]$, $[\Box^5_E]$, $[\Box^{\mathsf{IB4}}_E]$ and $[\Box^{\mathsf{IS5}}_E]$ are proved in the same way by respectively using $[\Box^B_L]$, $[\Box^4_L]$, $[\Box^5_L]$, $[\Box^{\mathsf{IB4}}_L]$ and $[\Box^{\mathsf{IS5}}_L]$. ■

In this proof of completeness, we use the normalization property satisfied by ND_{IKTh} to prove that if a T-sequent has a proof in ND_{IKTh}, then it has a proof in G_{IKTh} without any use of $[Cut]$.

Theorem 5 (Cut-elimination). *If a T-sequent has a proof in G_{IKTh}, then it has a proof in G_{IKTh} without the $[Cut]$ rule.*

Proof. Consequence of the normalization property in ND_{IKTh}. ■

We will give in the next section a direct proof of cut-elimination by proving by structural induction that the $[Cut]$ rule is admissible in the G^{-}_{IKTh} system. But before we aim at emphasizing the consequences of the cut-elimination property, namely the subformula property and also the depth property.

Theorem 6 (Subformula property). *If $\mathcal{S}$ is a T-sequent valid in* IKTh*, then there exists a proof of $\mathcal{S}$ in G^-_{IKTh} containing only subformulae of the formulae appearing in $\mathcal{S}$.*

In the case of G_{IKTh} for $\mathsf{Th} \in \{\emptyset, \{T\}, \{B\}, \{T, B\}\}$, there is also an important property that can be deduced from the subformula property. It concerns the depth of the T-sequents in a cut-free proof.

Proposition 3 (Depth property). *Let $\mathcal{S}$ be a T-sequent and $\mathcal{D}$ a proof of $\mathcal{S}$ in G^-_{IKTh} for $\mathsf{Th} \in \{\emptyset, \{T\}, \{B\}, \{T, B\}\}$. If $\mathcal{S}'$ is a T-sequent in $\mathcal{D}$ then its depth is less or equal to $d(\mathcal{S}) + nest(\mathcal{S})$.*

Proof. Let $\mathcal{S}'$ be a sequent in $\mathcal{D}$. By the subformula property, $\mathcal{S}'$ contains only subformulae of formulae in $\mathcal{S}$. Let A be a formula in $\mathcal{S}$ and B be an occurrence of a subformula of A in $\mathcal{S}'$. We write $l(B, \mathcal{S}')$ (resp. $l(A, \mathcal{S})$) to denote the length of the subbranch between B (resp. A) and the root of $\mathcal{S}'$ (resp. $\mathcal{S}$). Since G^-_{IKTh} does not contain the rules associated to 4, 5, $\mathsf{IB4}$ and $\mathsf{IS5}$, $l(B, \mathcal{S}') \leqslant l(A, \mathcal{S}) + nest(A)$. Indeed, this property can be shown by structural induction on $\mathcal{D}$. Let us consider, for instance, that the last rule application in $\mathcal{D}$ is:

$$\frac{\Gamma\{\langle A^{\vdash}\rangle\}}{\Gamma\{\Box A^{\vdash}\}}\ [\Box_R]$$

By induction hypothesis and knowing that $l(\Box A, \mathcal{S}) = l(A, \Gamma\{\langle A^{\vdash}\rangle\}) + 1$ and $nest(\Box A) = nest(A) + 1$, all subformula occurrences of the formulae in $\mathcal{S}$ in any T-sequent in $\mathcal{D}$ satisfy this property. Therefore, since for all formulae A in $\mathcal{S}$ and for all subformula occurrences B of A in $\mathcal{S}'$, we have $l(B, \mathcal{S}') \leqslant l(A, \mathcal{S}) + nest(A)$, we deduce that the depth of $\mathcal{S}'$ is smaller than or equal to $d(\mathcal{S}) + nest(\mathcal{S})$. Indeed, we have $nest(A) \leqslant nest(\mathcal{S})$ and $l(A, \mathcal{S}) \leqslant d(\mathcal{S})$. ■

The depth property fails in the case of G_{IS5} as illustrated by the following example:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\vdots}{\Gamma\{\Box\Diamond A, \Diamond A, \langle A, \Diamond A, \langle A, \Diamond A, \langle A\rangle\rangle\rangle\}}}{\Gamma\{\Box\Diamond A, \Diamond A, \langle A, \Diamond A, \langle A, \Diamond A\rangle\rangle\}}\ [\Diamond_L]}{\Gamma\{\Box\Diamond A, \langle A, \Diamond A, \langle A\rangle\rangle\}}\ [\Box_L^{\mathsf{IS5}}]}{\Gamma\{\Box\Diamond A, \Diamond A, \langle A, \Diamond A\rangle\}}\ [\Diamond_L]}{\Gamma\{\Box\Diamond A, \Diamond A, \langle A\rangle\}}\ [\Box_L^{\mathsf{IS5}}]}{\cfrac{\Gamma\{\Box\Diamond A, \Diamond A\}}{\Gamma\{\Box\Diamond A\}}\ [\Box_L^{\mathsf{IS5}}]}\ [\Diamond_L]$$

This property is not satisfied in calculi like G_{IS5} because of the left rules associated to $\Box$ and the right rules associated to $\Diamond$ for which there is no value that bounds the depth differences between the position of the marked formula in the premise and its position in the conclusion.

6 Structural cut-elimination in G_{IKTh}

In this section, we prove by structural induction that the cut rule is admissible in G^{-}_{IKTh}. We write $\vdash_G \mathcal{S}$ if the T-sequent $\mathcal{S}$ has a proof in the calculus G. Moreover, we write $\vdash^n_G \mathcal{S}$ if $\mathcal{S}$ has a proof in G of size smaller or equal to n.
Let us recall the notion of *size-preserving admissibility.*

Definition 5. *A rule $[R]$ is said to be* admissible *for a calculus G, if for all instances* $\dfrac{H_1 \ \ldots \ H_k}{C}\ [R]$ *of $[R]$, if we have $\vdash_G H_1, \ldots, \vdash_G H_k$ then $\vdash_G C$.*
A rule $[R]$ is said to be size-preserving admissible *for G, if for all n, if we have $\vdash^n_G H_1, \ldots, \vdash^n_G H_k$ then $\vdash^n_G C$.*

Now we prove that contraction, weakening and merge rules are size-preserving admissible in G^{-}_{IKTh}. They correspond to the following rules:

$$\dfrac{\Gamma\{\Delta, \Delta\}}{\Gamma\{\Delta\}}\ [C] \quad \dfrac{\Gamma\{C^{\vdash}\}\{\emptyset\}}{\Gamma\{C^{\vdash}\}\{A\}}\ [W] \quad \dfrac{\Gamma\{\langle\Delta\rangle, \langle\Delta'\rangle\}}{\Gamma\{\langle\Delta, \Delta'\rangle\}}\ [M]$$

Proposition 4 (Size-preserving admissibility of weakening). *For all $\mathsf{Th} \subseteq \{T, B, 4, 5\}$, if $\vdash^n_{G^{-}_{\mathsf{IKTh}}} \Gamma\{\emptyset\}\{C^{\vdash}\}$ then $\vdash^n_{G^{-}_{\mathsf{IKTh}}} \Gamma\{\Delta\}\{C^{\vdash}\}$.*

Proof. By induction on n. ∎

Proposition 5 (Size-preserving admissibility of merge). *For all $\mathsf{Th} \subseteq \{T, B, 4, 5\}$, if $\vdash^n_{G^{-}_{\mathsf{IKTh}}} \Gamma\{\langle\Delta\rangle, \langle\Delta'\rangle\}$ then $\vdash^n_{G^{-}_{\mathsf{IKTh}}} \Gamma\{\langle\Delta, \Delta'\rangle\}$.*

Proof. By induction on n. ∎

Proposition 6 (Size-preserving admissibility of contraction). *For all $\mathsf{Th} \subseteq \{T, B, 4, 5\}$, if $\vdash^n_{G^{-}_{\mathsf{IKTh}}} \Gamma\{\Delta, \Delta\}$ then $\vdash^n_{G^{-}_{\mathsf{IKTh}}} \Gamma\{\Delta\}$.*

Proof. Using Proposition 5, we only need to show the following property: (Pc) for all formula A, if $\vdash^n_{G^{-}_{\mathsf{IKTh}}} \Gamma\{A, A\}$ then $\vdash^n_{G^{-}_{\mathsf{IKTh}}} \Gamma\{A\}$. We can show it by induction on n. Indeed, Proposition 5 allows us to transform the case of $\Gamma\{\langle\Delta\rangle, \langle\Delta, \Delta'\rangle\}$ into the case of $\Gamma\{\langle\Delta, \Delta, \Delta'\rangle\}$ without increasing the size.
Let us consider the following case:

$$\frac{\Gamma\{\langle\Delta\rangle\}}{\Gamma\{\Delta\}}\,[SR^T] \qquad \frac{\Gamma\{\langle\Delta,\langle\Delta'\rangle\rangle\}}{\Gamma\{\langle\Delta\rangle,\Delta'\}}\,[SR^B]$$

$$\frac{\Gamma\{\Gamma'\{\emptyset\},\langle\Delta\rangle\}}{\Gamma\{\Gamma'\{\Delta\}\}}\,[SR^4](\dagger) \qquad \frac{\Gamma\{\langle\Delta\rangle\}\{\emptyset\}}{\Gamma\{\emptyset\}\{\Delta\}}\,[SR^5](\dagger\dagger)$$

$$\frac{\Gamma\{\langle\Delta\rangle\}\{\emptyset\}}{\Gamma\{\emptyset\}\{\Delta\}}\,[SR^{\mathsf{IB4}}](\dagger\dagger\dagger) \qquad \frac{\Gamma\{\langle\Delta\rangle\}\{\emptyset\}}{\Gamma\{\emptyset\}\{\Delta\}}\,[SR^{\mathsf{IS5}}]$$

($\dagger$) $depth(\Gamma'\{\}) > 1$;
($\dagger\dagger$) $depth(\Gamma\{\}\{\emptyset\}) > 0$ *and* $depth(\Gamma\{\emptyset\}\{\}) > 0$;
($\dagger\dagger\dagger$) $sp(\Gamma\{\emptyset\}\{\Delta\})$.

Fig. 5. Structural rules for G_{IKTh}

$$\dfrac{\begin{matrix}\mathcal{D}'\\ \Gamma\{\langle\Delta,\langle\Delta'\rangle\rangle,\langle\Delta,\langle\Delta'\rangle,C^{\vdash}\rangle\}\end{matrix}}{\Gamma\{\langle\Delta,\langle\Delta'\rangle\rangle,\langle\Delta,\langle\Delta'\rangle\rangle,\Diamond C^{\vdash}\}}\,[\Diamond_R]$$

where Δ and Δ' are multisets of formulae.

A proof of $\Gamma\{\langle\Delta,\langle\Delta'\rangle\rangle,\Diamond C^{\vdash}\}$ can be built using Property (Pc) as follows:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}'\\ \Gamma\{\langle\Delta,\langle\Delta'\rangle\rangle,\langle\Delta,\langle\Delta'\rangle,C^{\vdash}\rangle\}\end{matrix}}{\Gamma\{\langle\Delta,\Delta,\langle\Delta',\Delta'\rangle,C^{\vdash}\rangle\}}\,[M^2]}{\Gamma\{\langle\Delta,\langle\Delta'\rangle,C^{\vdash}\rangle\}}\,[(Pc)]}{\Gamma\{\langle\Delta,\langle\Delta'\rangle\rangle,\Diamond C^{\vdash}\}}\,[\Diamond_R]$$

■

The structural rules of Figure 5 are admissible in their corresponding logics.

Proposition 7. *Let* $\mathsf{Th} \subseteq \{T,B,4,5\}$ *such that* $\mathsf{Th} \neq \emptyset$*. Then, the following properties are satisfied:*

- *if* IKTh *is* $\mathsf{IS5}$*, then if* $[SR^{\mathsf{IS5}}]$ *is size-preserving admissible in* G^-_{IKTh}*;*
- *if* IKTh *is* $\mathsf{IB4}$*, then if* $[SR^{\mathsf{IB4}}]$ *is size-preserving admissible in* G^-_{IKTh}*;*
- *Otherwise, the rules of the form* $[SR^x]$ *such that* $x \in \mathsf{Th}$ *are size-preserving admissible in* G^-_{IKTh}*.*

Proof. Let us consider the case of the structural rule $[SR^T]$, the other cases being similar. We assume that we have $\vdash^n_{G^-_{\mathsf{IKTh}}} \Gamma\{\langle\Delta\rangle\}$ by a derivation $\mathcal{D}$. The

proof is by induction on n.
If $n = 1$ then $\Gamma\{\langle\Delta\rangle\}$ is an instance of an axiom and $\Gamma\{\Delta\}$ is consequently an instance of the same axiom. Thus, we have $\vdash^{1}_{G^{-}_{\mathsf{IKTh}}} \Gamma\{\Delta\}$. The proof is completed by mainly applying the induction hypothesis on the premise of the last rule application in $\mathcal{D}$ and using the rules $[\Diamond^{T}_{R}]$ and $[\Box^{T}_{L}]$. Let us consider, for instance, the case where $\Gamma\{\langle\Delta\rangle\} = \Gamma'\{\langle\Delta\rangle, \Diamond A^{\vdash}\}$ and the last rule application is the following:

$$\frac{\Gamma'\{\langle\Delta, A^{\vdash}\rangle\}}{\Gamma'\{\langle\Delta\rangle, \Diamond A^{\vdash}\}}\ [\Diamond_R]$$

By induction hypothesis, we have $\vdash^{n-1}_{G^{-}_{\mathsf{IKTh}}} \Gamma'\{\Delta, A^{\vdash}\}$. Then, by using the rule $[\Diamond^{T}_{R}]$ we obtain $\vdash^{n}_{G^{-}_{\mathsf{IKTh}}} \Gamma'\{\Delta, \Diamond A^{\vdash}\}$. ■

Theorem 7. *For all* $\mathsf{Th} \subseteq \{T, B, 4, 5\}$*, if* $\Gamma\{A^{\vdash}\}\{\emptyset\}$ *and* $\Gamma\{A\}\{C^{\vdash}\}$ *have proofs in* G^{-}_{IKTh}*, then* $\Gamma\{\emptyset\}\{C^{\vdash}\}$ *has also a proof in* G^{-}_{IKTh}*.*

Proof. We assume that we have $\vdash^{m}_{G^{-}_{\mathsf{IKTh}}} \Gamma\{A^{\vdash}\}\{\emptyset\}$ with a derivation $\mathcal{D}$ and $\vdash^{n}_{G^{-}_{\mathsf{IKTh}}} \Gamma\{A\}\{C^{\vdash}\}$ with a derivation $\mathcal{D}'$. We show that $\Gamma\{\emptyset\}\{C^{\vdash}\}$ has a proof in G^{-}_{IKTh} by mutual induction on m, n and the size of the cut-formula.

- If $m = 1$ then $\Gamma\{A^{\vdash}\}\{\emptyset\}$ is an instance of $[id]$ or $[\bot_L]$. In the case of the former, $\Gamma\{\emptyset\}\{C^{\vdash}\}$ has a proof in G^{-}_{IKTh}, since $\Gamma\{A\}\{C^{\vdash}\}$ has proof in G^{-}_{IKTh}. In the case of the latter, $\Gamma\{\emptyset\}\{C^{\vdash}\}$ is an instance of $[\bot_L]$ and has consequently a proof in G^{-}_{IKTh}.

- If $n = 1$ then $\Gamma\{A\}\{C^{\vdash}\}$ is an instance of $[id]$ or $[\bot_L]$. If A is not a principal formula in $\Gamma\{A\}\{C^{\vdash}\}$, then $\Gamma\{\emptyset\}\{C^{\vdash}\}$ is an instance of an axiom. Otherwise $\Gamma\{\emptyset\}\{C^{\vdash}\}$ has a proof in G^{-}_{IKTh}, since $\Gamma\{A^{\vdash}\}\{\emptyset\}$ has a proof in G^{-}_{IKTh}.

- If $\Gamma\{A\}\{C^{\vdash}\}$ is not an instance of an axiom and A is not a principal formula in the last rule application in $\mathcal{D}'$. We consider the case where this last rule application is an instance of a rule with two premises, the other case with one premise being similar:

$$\frac{\Gamma'\{A\}\{D^{\vdash}\} \qquad \Gamma''\{A\}\{E^{\vdash}\}}{\Gamma\{A\}\{C^{\vdash}\}}\ [R]$$

By the property of size-preserving admissibility of weakening (Proposition 4), there exist T-contexts $\Gamma^{1}_{w}\{\emptyset\}\{\emptyset\}$ and $\Gamma^{2}_{w}\{\emptyset\}\{\emptyset\}$ such that $\vdash^{n-1}_{G^{-}_{\mathsf{IKTh}}} \Gamma^{1}_{w}\{A\}\{D^{\vdash}\}$, $\vdash^{n-1}_{G^{-}_{\mathsf{IKTh}}} \Gamma^{2}_{w}\{A\}\{E^{\vdash}\}$, $\vdash^{m}_{G^{-}_{\mathsf{IKTh}}} \Gamma^{1}_{w}\{A^{\vdash}\}\{\emptyset\}$, $\vdash^{m}_{G^{-}_{\mathsf{IKTh}}} \Gamma^{2}_{w}\{A^{\vdash}\}\{\emptyset\}$ and then we have

$$\frac{\Gamma^1_w\{\emptyset\}\{D^\vdash\} \qquad \Gamma^2_w\{\emptyset\}\{E^\vdash\}}{\Gamma_w\{\emptyset\}\{C^\vdash\}}\ [R]$$

where $\Gamma_w\{\emptyset\}\{\emptyset\}$ is $\Gamma\{\emptyset\}\{\emptyset\}$ if C is a principal formula in $[R]$ or $[R] = [\supset_L]$, otherwise, it is obtained from $\Gamma\{\emptyset\}\{\emptyset\}$ by duplicating the principal formula of $[R]$. By induction hypothesis, we deduce that $\Gamma_w\{\emptyset\}\{C^\vdash\}$ has a proof in G^-_{IKTh}:

$$\frac{\dfrac{\Gamma^1_w\{A^\vdash\}\{\emptyset\} \quad \Gamma^1_w\{A\}\{D^\vdash\}}{\Gamma^1_w\{\emptyset\}\{D^\vdash\}}\,[I.H.] \qquad \dfrac{\Gamma^2_w\{A^\vdash\}\{\emptyset\} \quad \Gamma^2_w\{A\}\{E^\vdash\}}{\Gamma^2_w\{\emptyset\}\{E^\vdash\}}\,[I.H.]}{\Gamma_w\{\emptyset\}\{C^\vdash\}}\,[R]$$

By the property of size-preserving admissibility of contraction (Proposition 6), we deduce that $\Gamma\{\emptyset\}\{C^\vdash\}$ has a proof in G^-_{IKTh}.

- In the case where $\Gamma\{A^\vdash\}\{\emptyset\}$ is not an instance of an axiom and A is not a principal formula in the last rule application in $\mathcal{D}$, the proof is obtained from Proposition 4 and Proposition 6 in the same way as in the previous case.

- The case where A is an atomic formula, i.e., $A \in \mathsf{Prop}$ or $A = \bot$, is included in the previous cases.

- If $\Gamma\{A^\vdash\}\{\emptyset\}$ and $\Gamma\{A\}\{C^\vdash\}$ are not instances of axioms and A is a principal formula in the last rule applications of both $\mathcal{D}$ and $\mathcal{D}'$.
We consider the case of the rules associated to the modal connective $\Box$, the case for the modal connective $\Diamond$ being similar. The proof for the rules associated to the connectives $\supset$, $\wedge$ and $\vee$ is simpler and can be done in a standard way (e.g. see [29]).

Let us first consider the case where the last rule application in $\mathcal{D}$ is $[\Box_R]$ and that in $\mathcal{D}'$ is $[\Box_L]$:

$$\frac{\Gamma'\{\langle\Delta'\rangle, \langle A^\vdash\rangle\}}{\Gamma'\{\langle\Delta'\rangle, \Box A^\vdash\}}\ [\Box_R] \qquad \frac{\Gamma'\{\langle\Delta, A\rangle, \Box A\}}{\Gamma'\{\langle\Delta\rangle, \Box A\}}\ [\Box_L]$$

where $\Gamma'\{\langle\Delta\rangle\}\{\emptyset\} = \Gamma\{\emptyset\}\{\emptyset\}$ and $\Delta' = \Delta$ if Δ does not contain a marked formula, and $\Delta' = \Delta''\{\emptyset\}$ if $\Delta = \Delta''\{C^\vdash\}$.
By using the property of size-preserving admissibility of weakening (Proposition 4), we obtain $\vdash^m_{G^-_{\mathsf{IKTh}}} \Gamma'\{\langle\Delta', A\rangle, \Box A^\vdash\}$. By the induction hypothesis, we have $\vdash_{G^-_{\mathsf{IKTh}}} \Gamma'\{\langle\Delta, A\rangle\}$:

$$\frac{\Gamma'\{\langle\Delta', A\rangle, \Box A^\vdash\} \qquad \Gamma'\{\langle\Delta, A\rangle, \Box A\}}{\Gamma'\{\langle\Delta, A\rangle\}}\ [I.H.]$$

Then, by using the admissibility of the merge rule (Proposition 5), we deduce that $\vdash_{G^-_{\mathsf{IKTh}}} \Gamma'\{\langle \Delta', A^{\vdash}\rangle\}$. Thus, by induction hypothesis (A is smaller than $\Box A$, i.e., $|A| < |\Box A|$) with $\Gamma'\{\langle \Delta', A^{\vdash}\rangle\}$ and $\Gamma'\{\langle \Delta, A\rangle\}$ as premises, we deduce that $\vdash_{G^-_{\mathsf{IKTh}}} \Gamma'\{\langle \Delta\rangle\}$.

Let us now consider the case where the last rule application in $\mathcal{D}$ is $[\Box_R]$ and that in $\mathcal{D}'$ is $[\Box^T_L]$:

$$\frac{\Gamma\{\langle A^{\vdash}\rangle\}\{\emptyset\}}{\Gamma\{\Box A^{\vdash}\}\{\emptyset\}}\ [\Box_R] \qquad \frac{\Gamma\{A, \Box A\}\{C^{\vdash}\}}{\Gamma\{\Box A\}\{C^{\vdash}\}}\ [\Box^T_L]$$

By using the property of size-preserving admissibility of weakening and the induction hypothesis, we have $\vdash_{G^-_{\mathsf{IKTh}}} \Gamma\{A\}\{C^{\vdash}\}$:

$$\frac{\dfrac{\Gamma\{\Box A^{\vdash}\}\{\emptyset\}}{\Gamma\{A, \Box A^{\vdash}\}\{\emptyset\}}\ [W] \qquad \Gamma\{A, \Box A\}\{C^{\vdash}\}}{\Gamma\{A\}\{C^{\vdash}\}}\ [I.H.]$$

From the property of size-preserving admissibility of $[SR^T]$ (Proposition 7), we obtain $\vdash_{G^-_{\mathsf{IKTh}}} \Gamma\{A^{\vdash}\}\{\emptyset\}$. Moreover, since $|A| < |\Box A|$, one can apply the induction hypothesis with $\Gamma\{A^{\vdash}\}\{\emptyset\}$ and $\Gamma\{A\}\{C^{\vdash}\}$ as premises:

$$\frac{\Gamma\{A^{\vdash}\}\{\emptyset\} \qquad \Gamma\{A\}\{C^{\vdash}\}}{\Gamma'\{\emptyset\}\{C^{\vdash}\}}\ [I.H.]$$

Therefore, we deduce that $\vdash_{G^-_{\mathsf{IKTh}}} \Gamma'\{\emptyset\}\{C^{\vdash}\}$.

The proof for the cases where the last rule application in $\mathcal{D}'$ are $[\Box^B_L]$, $[\Box^4_L]$, $[\Box^{\mathsf{IB4}}_L]$ and $[\Box^{\mathsf{IS5}}_L]$ is obtained in the same way by using the induction hypothesis and the admissibility of respectively $[SR^B]$, $[SR^4]$, $[SR^{\mathsf{IB4}}]$ and $[SR^{\mathsf{IS5}}]$. ■

Then we have the following result:

Theorem 8. *The cut rule is admissible in G^-_{IKTh}.*
Thus if a T-sequent has a proof in G_{IKTh}, then it has a proof in G_{IKTh} without the [cut] rule.

7 Proof-search and decidability

In this section, we propose new decision procedures using the calculi for the logics IK, IT, $\mathsf{IK}B$ and $\mathsf{IK}TB$. We restrict our study to these logics because they have the depth property that is necessary to prove termination within our approach.

In this perspective we introduce a notion of *redundancy* in cut-free derivations for the calculi in order to have the following result: any T-sequent that is valid has a proof without redundancies. Then, by using the subformula property and the depth property, we show that there is no infinite proof which is not irredundant and then we provide some decision procedures based on this result.

7.1 Redundant derivations

We first extend the notion of contraction which allows us to capture the notion of merging.

Proposition 8. *The following rule is size-preserving admissible:*

$$\frac{\Gamma\{\langle\Delta, C^{\vdash}\rangle, \langle\Delta\rangle\}}{\Gamma\{\langle\Delta, C^{\vdash}\rangle\}}\ [C_2]$$

Proof. By using the property of size-preserving admissibility of contraction and merge, we have

$$\dfrac{\dfrac{\Gamma\{\langle\Delta, C^{\vdash}\rangle, \langle\Delta\rangle\}}{\Gamma\{\langle\Delta, \Delta, C^{\vdash}\rangle\}}\,[M]}{\Gamma\{\langle\Delta, C^{\vdash}\rangle\}}\,[C]$$

■

We define the two relations $\to_c$ (contraction), $\to_w$ (weakening) by $\Gamma\{\Delta, \Delta\} \to_c \Gamma\{\Delta\}$, $\Gamma\{\langle\Delta, C^{\vdash}\rangle, \langle\Delta\rangle\} \to_c \Gamma\{\langle\Delta, C^{\vdash}\rangle\}$ and $\Gamma\{C^{\vdash}\}\{\emptyset\} \to_w \Gamma\{C^{\vdash}\}\{\Sigma\}$ (Σ is T-context).

Moreover, we define the preorder relation on the T-sequents $\lesssim$ by: $\mathcal{S} \lesssim \mathcal{S}'$ if and only if $\mathcal{S}(\to_c + \to_w)^*\mathcal{S}'$ where $(\to_c + \to_w)^*$ is the reflexive and transitive closure of the union of the two relations $\to_c$ and $\to_w$. We denote $\cong$ the equivalence relation defined by: $\mathcal{S} \cong \mathcal{S}'$ if and only if $\mathcal{S} \lesssim \mathcal{S}'$ and $\mathcal{S}' \lesssim \mathcal{S}$.

Proposition 9. *Let $\mathcal{S}$ and $\mathcal{S}'$ be T-sequents such that $\mathcal{S} \lesssim \mathcal{S}'$. If $\vdash^n_{G^-_{\mathsf{IKTh}}} \mathcal{S}$ then $\vdash^n_{G^-_{\mathsf{IKTh}}} \mathcal{S}'$.*

Proof. Direct consequence of the size-preserving admissibility of weakening ($[W]$) and contraction($[C]$ and $[C_2]$). ■

Let us note that the rule $[M]$ can be obtained from the rules $[W]$, $[C]$ and $[C_2]$. Indeed, assuming that we have $\vdash_{G^-_{\mathsf{IKTh}}} \Gamma\{\langle\Delta_1\rangle, \langle\Delta_2\rangle\}$, if Δ_1 and Δ_2 do not contain a marked formula then $[M]$ is obtained as follows:

$$\dfrac{\dfrac{\Gamma\{\langle\Delta_1\rangle, \langle\Delta_2\rangle\}}{\Gamma\{\langle\Delta_1, \Delta_2\rangle, \langle\Delta_1, \Delta_2\rangle\}}\,[W]}{\Gamma\{\langle\Delta_1, \Delta_2\rangle\}}\,[C]$$

Otherwise, if Δ_1 contains a marked formula ($\Delta_1 = \Delta'_1, C^{\vdash}$):

$$\dfrac{\dfrac{\Gamma\{\langle \Delta'_1, C^{\vdash}\rangle, \langle \Delta_2\rangle\}}{\Gamma\{\langle \Delta'_1, \Delta_2, C^{\vdash}\rangle, \langle \Delta'_1, \Delta_2\rangle\}}\,[W]}{\Gamma\{\langle \Delta'_1, \Delta_2, C^{\vdash}\rangle\}}\,[C_2]$$

The case where Δ_2 contains a marked formula is similar to the previous case.

Definition 6 (Redundant derivation). *A derivation is said to be* redundant *if it contains two T-sequents $\mathcal{S}_1$ and $\mathcal{S}_2$, with $\mathcal{S}_1$ occurring strictly above $\mathcal{S}_2$ in the same branch, such that $\mathcal{S}_1 \lesssim \mathcal{S}_2$. It is said to be* irredundant *if it is not redundant.*

Proposition 10. *For all $\mathsf{Th} \subseteq \{T, B, 4, 5\}$, if a T-sequent is valid in IKTh then it has an irredundant proof in G^-_{IKTh}.*

Proof. Let $\mathcal{S}$ be a T-sequent and $\mathcal{D}$ be a proof of $\mathcal{S}$ in G^-_{IKTh}. This proof is by induction on the size s of $\mathcal{D}$.

- If $s = 1$ then it is an irredundant proof.

- We assume that for any T-sequent, if it has a proof of size smaller or equal to n ($n \geqslant 1$), then it has a irredundant proof (induction hypothesis). We assume that $s = n + 1$.
If $\mathcal{D}$ is irredundant then it has an irredundant proof. Otherwise it has a branch containing two T-sequents $\mathcal{S}_1$ and $\mathcal{S}_2$ such that $\mathcal{S}_1$ occurring above $\mathcal{S}_2$ and $\mathcal{S}_1 \lesssim \mathcal{S}_2$. Let n' be the size of the sub-derivation of $\mathcal{S}_1$ in $\mathcal{D}$. We can see that the size of the sub-derivation of $\mathcal{S}_2$ in $\mathcal{D}$ is strictly greater than n'. Using Proposition 9, we know that $\mathcal{S}_2$ has a proof $\mathcal{D}_2$ of size smaller or equal to n'. Then, by replacing in $\mathcal{D}$ the sub-derivation of $\mathcal{S}_2$ with $\mathcal{D}_2$, we obtain a derivation of $\mathcal{S}$ of size smaller or equal to n. Therefore, by applying the induction hypothesis, we can deduce that $\mathcal{S}$ has a irredundant proof. ■

7.2 New decision procedures

Let us now introduce the key point of our proof of termination. Since we need the depth property (Proposition 3) the following proposition is true only in the cases of G_{IKTh} with $\mathsf{Th} \in \{\emptyset, \{T\}, \{B\}, \{T, B\}\}$.

Proposition 11. *Let $\mathcal{S}$ be a T-sequent, $\mathsf{Th} \in \{\emptyset, \{T\}, \{B\}, \{T, B\}\}$ and $\mathcal{D}$ be a derivation of $\mathcal{S}$ in G^-_{IKTh}. The set of all T-sequents appearing in $\mathcal{D}$ is partitioned into a finite set of equivalence classes by $\cong$.*

Proof. It is a consequence of the subformula property (Proposition 6) and the depth property (Proposition 3). Indeed, using these two properties, we know that the set $\mathcal{B}$ of all branches of all T-sequents appearing in $\mathcal{D}$ is finite. Let us assume that $\mathcal{D}$ contains an infinite set $\mathcal{E}$ of T-sequents that are pairwise inequivalent. Since $\mathcal{B}$ is finite, we can deduce that there exist two T-sequents $\mathcal{S}_1$ and $\mathcal{S}_2$ in $\mathcal{E}$ such that $\mathcal{S}_1 \rightarrow_c^* \mathcal{S}_2$ ($\mathcal{S}_1 \lesssim \mathcal{S}_2$) and $\mathcal{S}_2 \rightarrow_w^* \mathcal{S}_1$ ($\mathcal{S}_2 \lesssim \mathcal{S}_1$). Thus, we have $\mathcal{S}_1 \cong \mathcal{S}_2$ and we get a contradiction. ■

We give now a decision procedure for IKTh with $\mathsf{Th} \in \{\emptyset, \{T\}, \{B\}, \{T, B\}\}$, that is based on the related G^-_{IKTh} calculus and on a search of an irredundant proof of the given T-sequent.

A Decision procedure for IKTh ($\mathsf{Th} \in \{\emptyset, \{T\}, \{B\}, \{T, B\}\}$) :

Let $\mathcal{S}$ be a T-sequent.

- Step 1. We start with the derivation containing only $\mathcal{S}$ which is the unique irredundant derivation of size 1. If this derivation is a proof then we return it. Otherwise we move to the next step.

- Step $i+1$. We build the set of all the irredundant derivations of size $i+1$. If this set contains a proof of $\mathcal{S}$ then we return it. Otherwise if this set is empty then $\mathcal{S}$ is not valid, else we move to the next step.

The correctness of this algorithm is obtained from the soundness and completeness of the sequent calculi and from Proposition 10.
Regarding the termination of the algorithm, it is a consequence of Proposition 11 and of the fact that there is only a finite number of possible rule applications allowing us to extend the size from i to $i+1$.

Such a procedure provides also a new syntactic proof of decidability of the corresponding logic.

Theorem 9. *For all* $\mathsf{Th} \in \{\emptyset, \{T\}, \{B\}, \{T, B\}\}$*, the logic* IKTh *is decidable.*

In the following example, we use our decision procedure for IT in order to prove that the formula $\Diamond\Box A \supset A$ is not valid.

Let us denote by Der_i the set of the irredundant derivations of size i.

Step 1: $Der_1 \equiv \Diamond\Box A \supset A^{\vdash}$

Step 2: $Der_2 \equiv \dfrac{\Diamond\Box A, A^{\vdash}}{\Diamond\Box A \supset A^{\vdash}}\,[\supset_R]$

Step 3: $Der_3 \equiv \dfrac{\dfrac{\Diamond\Box A, \langle\Box A\rangle, A^{\vdash}}{\Diamond\Box A, A^{\vdash}}\,[\Diamond_L]}{\Diamond\Box A \supset A^{\vdash}}\,[\supset_R]$

Step 4: $Der_4 \equiv$

$$\dfrac{\dfrac{\dfrac{\Diamond\Box A, \langle \Box A, A\rangle, A^{\vdash}}{\Diamond\Box A, \langle \Box A\rangle, A^{\vdash}}\,[\Box_L^T]}{\Diamond\Box A, A^{\vdash}}\,[\Diamond_L]}{\Diamond\Box A \supset A^{\vdash}}\,[\supset_R]$$

The derivations of size 5 that we can obtain from the derivation in Der_4 are the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Diamond\Box A, \langle \Box A, A, A\rangle, A^{\vdash}}{\Diamond\Box A, \langle \Box A, A\rangle, A^{\vdash}}\,[\Box_L^T]}{\Diamond\Box A, \langle \Box A\rangle, A^{\vdash}}\,[\Box_L^T]}{\Diamond\Box A, A^{\vdash}}\,[\Diamond_L]}{\Diamond\Box A \supset A^{\vdash}}\,[\supset_R] \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Diamond\Box A, \langle \Box A\rangle, \langle \Box A, A\rangle, A^{\vdash}}{\Diamond\Box A, \langle \Box A, A\rangle, A^{\vdash}}\,[\Diamond_L]}{\Diamond\Box A, \langle \Box A\rangle, A^{\vdash}}\,[\Box_L^T]}{\Diamond\Box A, A^{\vdash}}\,[\Diamond_L]}{\Diamond\Box A \supset A^{\vdash}}\,[\supset_R]$$

All these derivations are redundant knowing that we have both:

- $\Diamond\Box A, \langle \Box A, A, A\rangle, A^{\vdash} \lesssim \Diamond\Box A, \langle \Box A, A\rangle, A^{\vdash}$
- $\Diamond\Box A, \langle \Box A\rangle, \langle \Box A, A\rangle, A^{\vdash} \lesssim \Diamond\Box A, \langle \Box A, A\rangle, A^{\vdash}$.

Therefore, we have $Der_5 = \emptyset$ and we deduce that $\Diamond\Box A \supset A$ is not valid in IT.

8 Conclusion and perspectives

In this paper we define label-free sequent calculi for the intuitionistic modal logics based on the combinations of the axioms T, B, 4 and 5. These calculi are based on the use of a particular multi-contextual structure, called Tree-sequent, introduced in [14] for defining natural deduction systems for such logics. We also show that our calculi satisfy the cut-elimination property and as consequences the subformula and depth properties. Moreover we propose alternative decision procedures for IK, IT, IKB and IKTB that are based on these calculi. We only consider these logics because the others do not satisfy the depth property that is required for proving termination in this approach.

Further work will be devoted to the design of label-free sequent calculi for other intuitionistic modal logics based on similar structures. Moreover, we will study the decidability of IS4 and IK4 which is still an open question [27] by using such sequent calculi.

References

1. A. Avron. The method of hypersequents in the proof theory of propositional non-classical logics. In W. Hodges, M. Hyland, C. Steinhorn, and J. Truss, editors, *Logic: from foundations to applications*, pages 1–32. Oxford University Press, 1996.

2. G. M. Bierman and V. de Paiva. On an intuitionistic modal logic. *Studia Logica*, 65:383–416, 2000.
3. K. Brünnler. Deep sequent systems for modal logic. *Arch. Math. Log.*, 48(6):551–577, 2009.
4. K. Brünnler and L. Straßburger. Modular Sequent Systems for Modal Logic. In *Int. Conference on Analytic Tableaux and Related Methods, TABLEAUX 2009, LNAI 5607*, pages 152–166, 2009. Oslo, Norway.
5. R. A. Bull. A modal extension of intuitionistic logic. *Notre Dame Journal of Formal Logic*, 6:142–145, 1965.
6. M. J. Collinson, B. Hilken, and D. Rydeheard. Semantics and proof theory of an intuitionistic modal sequent calculus. Technical Report UMCS-99-6-1, University of Manchester, UK, 1999.
7. R. Davies and F. Pfenning. A modal analysis of staged computation. *Journal of ACM*, 48(3):555–604, 2001.
8. M. Fairtlough and M. Mendler. An intuitionistic modal logic with applications to the formal verification of hardware. In *Conference on Computer Science Logic, CSL'94*, pages 354–368, Kazimierz, Poland, 1995.
9. G. Fischer-Servi. The finite model property for MIPQ and some consequences. *Notre Dame Journal of Formal Logic*, XIX:687–692, 1978.
10. G. Fischer-Servi. Semantics for a class of intuitionistic modal calculi. *Italian Studies in the Philosophy of Science*, pages 59–72, 1981.
11. G. Fischer-Servi. Axiomatizations for some intuitionistic modal logics. *Rend. Sem. Mat. Univers. Politecn. Torino*, 42:179–194, 1984.
12. M. Fitting. Prefixed Tableaus and Nested Sequents. *Annals of Pure and Applied Logic*, 163(3):291–313, 2012.
13. J. M. Font. Modality and possibility in some intuitionistic modal logics. *Notre Dame Journal of Formal Logic*, 27:533–546, 1986.
14. D. Galmiche and Y. Salhi. Label-free natural deduction systems for intuitionistic and classical modal logics. *Journal of Applied Non-Classical Logics*, 20(4):373–421, 2010.
15. D. Galmiche and Y. Salhi. Label-free proof systems for modal intuitionistic logic IS5. In *Int. Conference on Logic for Programming, Artificial Intelligence, and Reasoning, LPAR 2010, LNAI 6365*, pages 255–271, Dakar, Senegal, May 2010.
16. R. Goré, L. Postniece, and A Tiu. Taming Displayed Tense Logics using Nested Sequents with Deep Inference. In *Int. Conference on Analytic Tableaux and Related Methods, TABLEAUX 2009, LNAI 5607*, pages 205–219, 2009. Oslo, Norway.
17. R. Goré, L. Postniece, and A. Tiu. Cut-Elimination and Proof Search for Bi-Intuitionistic Tense Logic. In *Advances in Modal Logic, AiML 2010*, pages 39–48, 2010. Moscow, Russia.
18. L. Jia and D. Walker. Modal proofs as distributed programs (extended abstract). In *European Symposium on Programming, ESOP'04, LNCS 2986*, pages 219–233, Barcelona, Spain, 2004.
19. T. Murphy VII, K. Crary, R. Harper, and F. Pfenning. A symmetric modal lambda calculus for distributed computing. In *In Proceedings of the 19th IEEE Symposium on Logic in Computer Science (LICS)*, pages 286–295. IEEE Press, 2004.
20. S. Negri. Proof analysis in modal logic. *Journal of Philosophical Logic*, 34:507–534, 2005.
21. S. Negri. Proof theory for modal logic. *Philosophy Compass*, 6/8:523–538, 2011.
22. H. Ono. On some intuitionistic modal logics. *Publications of the Research Institute for Mathematical Science*, 13:55–67, 1977.

23. H. Ono and N. Suzuki. Relations between intuitionistic modal logics and intermediate predicate logics. *Reports on Mathematical Logic*, 22:65–87, 1988.
24. G. Plotkin and C. Stirling. A Framework for Intuitionistic Modal Logics. In *Theoretical Aspects of Reasoning about Knowledge, TARK'86*, pages 399–406, 1986. Monterey, CA.
25. F. Poggiolesi. The Method of Tree-Hypersequents for Modal Propositional Logic. In *Towards Mathematical Philosophy*, volume 28 of *Trends in Logic*, pages 31–51. Springer Netherlands, 2009.
26. D. Prawitz. *Natural Deduction: A Proof-Theoretical Study.* Almquist and Wiksell, Stockholm, 1965.
27. A. Simpson. *The proof theory and semantics of intuitionistic modal logic.* PhD thesis, University of Edinburgh, 1994.
28. N. Suzuki. An algebraic approach to intuitionistic modal logics in connection with intermediate predicate logics. *Studia Logica*, 48:141–155, 1990.
29. A.S. Troelstra and H. Schwichtenberg. *Basic Proof Theory*, volume 43 of *Cambridge Tracts in Theoretical Computer Science.* Cambridge University Press, 1996.
30. H. Wansing. *Displaying Modal Logic.* Kluwer Academic Publishers, 1998.