TICAMORE

Translating and discovering Calculi for Modal and Related logics

Bounded Sequent Calculi via Hypersequents

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FWF Der Wissenschaftsfonds.
logics LOGICAL METHODS IN COMPUTER SCIENCE

Introduction

$\mathcal{C}$: calculus for a logic L

$\mathcal{C}$ is analytic $:\Leftrightarrow$
Every $F \in L$ has a $\mathcal{C}$-proof using only subformulas of F

- decreased proof search space,
- But: Usefulness of analyticity is relative to complexity of $\mathcal{C}$
  MELL, **MTL** – analytic (hyper)sequent calculus ✓
  – terminating proof search ?
- For more complicated $\mathcal{C}$, analyticity is easier to obtain
- $\Rightarrow$ Want to find the simplest $\mathcal{C}$ which is analytic for L.

Sequent calculus

Gerhard Gentzen
(1909-1945)

Sequents

$$A_1, \dots, A_n \Rightarrow B_1, \dots, B_n \qquad A_1, \dots, A_n \Rightarrow B$$

Initial sequents

$$A \Rightarrow A$$

Logical rules, i.e.

$$\frac{\Gamma \Rightarrow A \quad \Delta, B \Rightarrow C}{\Gamma, A \to B, \Delta \Rightarrow C} \to_L \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B} \to_R$$

Cut

$$\frac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow B} \; cut$$

Logics with an analytic sequent calculus:

- ✓ classical logic (propositional and first-order)
- ✓ intuitionistic logic (propositional and first-order)
- ✓ modal logics: **K**, **S4**, **S5**(!)
- ✓ basic substructural logics: **FL**, **FLe**, **FLew**, **FLewc**, **FLec**

and without:

- X fuzzy logics: **G**,**Ł**,**P**
- X intermediate logics: Kripke models of bounded width/size
- X various substructural logics

Hypersequent calculus

Grigori Mints
(1939-2014)

Garrell Pottinger

Arnon Avron
(*1952)

Hypersequents

$$\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n$$

Initial (hyper)sequents

$$A \Rightarrow A$$

External structural rules

$$\frac{G}{G \mid \Gamma \Rightarrow \Pi}\ ew \qquad \frac{G \mid \Gamma \Rightarrow \Pi \mid \Gamma \Rightarrow \Pi}{G \mid \Gamma \Rightarrow \Pi}\ ec$$

Hyperrules, i.e.

$$\frac{G \mid \Gamma_1, \Delta_1 \Rightarrow \Pi_1 \quad G \mid \Gamma_2, \Delta_2 \Rightarrow \Pi_2}{G \mid \Gamma_1, \Delta_2 \Rightarrow \Pi_1 \mid \Gamma_2, \Delta_1 \Rightarrow \Pi_2}\ com$$

Logics with an analytic hypersequent calculus:

- ✓ fuzzy logics: **G**,**Ł**,**P**
- ✓ intermediate logics: Kripke models of bounded width/size
- ✓ various substructural logics:

Ciabattoni, Galatos and Terui [LICS08]

Every logic of the form

$$\mathbf{FLe}(\mathbf{wc}) + A$$

– where A is an amenable axioms – admits an analytic (in fact, cut-free) hypersequent calculus.

amenable = class $\mathcal{P}_3/\mathcal{P}_3'$ in the substructural hierarchy

End of Introduction

A. Ciabattoni, T. Lang and R. Ramanayake:
Bounded sequent calculi for non-classical logics via hypersequents
in TABLEAUX 2019

Guiding question:

> If L has an analytic hypersequent calculus,
> how far is L from being analytic in the sequent calculus?

Strategy:

- Define a notion weaker than analyticity ('boundedness')
- cutfree hypersequent proof $\stackrel{\text{proof transformation}}{\longrightarrow}$ bounded sequent proof
- Show that 'boundedness' is a useful property in the sequent calculus

logic = propositional substructural/intermediate
(later: modal)

$$
\begin{array}{ccc}
L & \xrightarrow{\textbf{axiomatic extension}} & L + A \\
\updownarrow & & \updownarrow \\
Seq_L & \longrightarrow & Seq_L + (\Rightarrow A)
\end{array}
$$

Cut is not admissible in $Seq_L + (\Rightarrow A)$!

$$\vdash_{Seq_L + (\Rightarrow A)} F \quad \text{iff} \quad \vdash^{cf}_{Seq_L} (A_1)_{\wedge 1}, \ldots, (A_n)_{\wedge 1} \Rightarrow F$$

for some list $(A_i)_i$ of A-instances

Can we predict $(A_i)_i$ from F?

$$\vdash_{Seq_L+(\Rightarrow A)} F \quad \text{iff} \quad \vdash^{cf}_{Seq_L} (A_1)_{\wedge 1}, \ldots, (A_n)_{\wedge 1} \Rightarrow F$$
for some list $(A_i)_i$ of A-instances

$L + A$ is ...

$\boxed{\text{variable-bounded}}$	each A_i is of the form
	$A(q) \qquad q \in \mathrm{Var}(F)$
$\boxed{\text{formula-bounded}}$	each A_i is of the form
	$A(F') \qquad F' \in \mathrm{subf}(F)$
$\boxed{\text{set-bounded}}$	each A_i is of the form
	$A(F_1 * \ldots * F_n) \qquad F_i \in \mathrm{subf}(F)$ and $F_i \neq F_j$
$\boxed{\text{multiset-bounded}}$	each A_i is of the form
	$A(F_1 * \ldots * F_n) \qquad F_i \in \mathrm{subf}(F)$

Remark: Boundedness is a property of the logic.

Example: $A = A(p)$, $F = r \to r * r$

- variable-bounded instances:
  $A(r)$
- formula-bounded instances:
  $\ldots, A(r * r), A(r \to r * r)$
- set-bounded instances:
  $\ldots, A(r * (r \to r * r)), A(r * (r * r)), A((r * r) * (r \to r * r)),$
  $A(r * (r * r) * (r \to r * r)), A(0)$
- multiset-bounded instances:
  $\ldots, A(r * r * r), A(r * r * r * r), A(r * r * r * r * r) \ldots$

Lemma

$L + A$ axiomatic extension, contraction admissible in L.

1. If mingle is admissible in L, then

$$L + A \text{ multiset-bounded} \quad \Leftrightarrow \quad L + A \text{ set-bounded}$$

2. If $L + A$ is set-bounded, then

$$L + A \overset{\text{exp-time}}{\to} L$$

3. If $L + A$ is formula-bounded, then

$$L + A \overset{\text{p-time}}{\to} L$$

   In particular $cc(L + A) \leq cc(L)$.

LEMMA 2. *Let A be a formula constructed with the propositional variables $a_1, a_2, \ldots, a_n$. Then, $A \in$ **LK** if and only if*

$$((a_1 \vee \neg a_1) \wedge (a_2 \vee \neg a_2) \wedge \ldots \wedge (a_n \vee \neg a_n)) \supset A \in \mathbf{LJ}.$$

Remark that the above lemma is not necessarily the immediate consequence of the fact $\mathbf{LK} = \mathbf{LJ} + a \vee \neg a$. Further, we remark here that the lemma gives a decision procedure for **LK** via **LJ**.

from *T.Hosoi: Pseudo two-valued evaluation method for intermediate logics* (1986)

$\Rightarrow$ $\mathbf{LJ} + p \vee \neg p$ is variable-bounded

Main result

Every amenable extension $\mathbf{FL_{e(wc)}} + A$ is multiset-bounded.

Example: *monoidal t-norm logic*

$\wedge, \vee, \rightarrow, *$ (fusion), $0, 1, \bot, \top$

$$\mathbf{MTL} = \mathbf{FLew} + \underbrace{(p \rightarrow q) \vee (q \rightarrow p)}_{\text{linearity axiom}}$$

Corresponding cutfree calculus:

$$h\mathbf{MTL} = h\mathbf{FLew} + \frac{G \mid \Gamma_1, \Delta_1 \Rightarrow \Pi_1 \quad G \mid \Gamma_2, \Delta_2 \Rightarrow \Pi_2}{G \mid \Gamma_1, \Delta_2 \Rightarrow \Pi_1 \mid \Gamma_2, \Delta_1 \Rightarrow \Pi_2}\ com$$

Assume $F \in \mathbf{FLew} + (p \to q) \vee (q \to p)$.

$$\dfrac{\overset{\delta}{G \mid \Gamma_1, \Delta_1 \Rightarrow \Pi_1} \quad \overset{\gamma}{G \mid \Gamma_2, \Delta_2 \Rightarrow \Pi_2}}{\dfrac{G \mid \Gamma_1, \Delta_2 \Rightarrow \Pi_1 \mid \Gamma_2, \Delta_1 \Rightarrow \Pi_2}{\vdots \atop \Rightarrow F}}\ com$$

LEFT SPLIT

$$\dfrac{\dfrac{\Delta_2 \Rightarrow *\Delta_2 \quad \overset{\delta}{G \mid \Gamma_1, *\Delta_1 \Rightarrow \Pi_1}}{G \mid \boxed{*\Delta_2 \to *\Delta_1}, \Gamma_1, \Delta_2 \Rightarrow \Pi_1}\ \to_L}{G \mid \boxed{*\Delta_2 \to *\Delta_1}, \Gamma_1, \Delta_2 \Rightarrow \Pi_1 \mid \Gamma_2, \Delta_1 \Rightarrow \Pi_2}\ ew$$

$$\vdots$$

$$\boxed{*\Delta_2 \to *\Delta_1} \Rightarrow F$$

The proof transformation (2/3)

$$\dfrac{\overset{\delta}{G \mid \Gamma_1, \Delta_1 \Rightarrow \Pi_1} \quad \overset{\gamma}{G \mid \Gamma_2, \Delta_2 \Rightarrow \Pi_2}}{G \mid \Gamma_1, \Delta_2 \Rightarrow \Pi_1 \mid \Gamma_2, \Delta_1 \Rightarrow \Pi_2}\ com$$

$$\vdots$$

$$\Rightarrow F$$

RIGHT SPLIT

$$\dfrac{\dfrac{\Delta_1 \Rightarrow *\Delta_1 \quad \overset{\gamma}{G \mid \Gamma_2, *\Delta_2 \Rightarrow \Pi_2}}{G \mid \boxed{*\Delta_1 \rightarrow *\Delta_2}, \Gamma_2, \Delta_1 \Rightarrow \Pi_2}\ \rightarrow_L}{G \mid \boxed{*\Delta_1 \rightarrow *\Delta_2}, \Gamma_2, \Delta_1 \Rightarrow \Pi_2 \mid \Gamma_1, \Delta_2 \Rightarrow \Pi_1}\ ew$$

$$\vdots$$

$$\boxed{*\Delta_1 \rightarrow *\Delta_2} \Rightarrow F$$

The proof transformation (3/3)

$$\dfrac{\overset{\text{LEFT SPLIT}}{\vdots} \qquad \overset{\text{RIGHT SPLIT}}{\vdots}}{\dfrac{\boxed{*\Delta_2 \to *\Delta_1} \Rightarrow F \qquad \boxed{*\Delta_1 \to *\Delta_2} \Rightarrow F}{\boxed{(*\Delta_2 \to *\Delta_1) \vee (*\Delta_1 \to *\Delta_2)} \Rightarrow F} \vee_L}$$

Original hypersequent proof was cutfree, hence:

- $\rightsquigarrow$ $\Delta_1, \Delta_2 \subseteq \mathrm{Subf}(F)$
- $\rightsquigarrow$ $(*\Delta_2 \to *\Delta_1) \vee (*\Delta_1 \to *\Delta_2)$ is an instantiation of the linearity axiom by fusions of subformulas of F

Iterate this!

For termination: remove all lowermost (*com*)'s **simultaneously**

$$(*\Delta^1_2 \to *\Delta^1_1) \vee (*\Delta^1_1 \to *\Delta^1_2), \ldots, (*\Delta^n_2 \to *\Delta^n_1) \vee (*\Delta^n_1 \to *\Delta^n_2) \Rightarrow F$$

$\Delta^i_j \subseteq \mathrm{Subf}(F)$

Proof in h**MTL** without (com) $\equiv$ Proof in **FLew**

Theorem (special case)

FLew $+ (p \to q) \vee (q \to p)$ is multiset-bounded.

The same approach works for all logics discussed in [LICS08]:

Theorem (general case)

Every amenable extension $\mathbf{FL_{e(wc)}} + A$ is multiset-bounded.

Corollaries

Lemma (Recall)

$L + A$ axiomatic extension, contraction admissible in L.

1. If mingle is admissible in L, then

$$L + A \text{ multiset-bounded} \quad \Leftrightarrow \quad L + A \text{ set-bounded}$$

2. If $L + A$ is set-bounded, then

$$L + A \stackrel{\text{exp-time}}{\to} L$$

3. If $L + A$ is formula-bounded, then

$$L + A \stackrel{\text{p-time}}{\to} L$$

   In particular $cc(L + A) \leq cc(L)$.

Corollary

Every amenable extension of **LJ** (actually: $\mathbf{FL_{ecm}}$) is exp-time reducible to **LJ** ($\mathbf{FL_{ecm}}$), and therefore has computational complexity $\leq 2EXP$.

Some ongoing work...

- Defined a syntactic property U such that:
  $\mathbf{LJ} + A$ set-bounded & $A \in U$ $\rightarrow$ $\mathbf{LJ} + A$ formula-bounded.

  Examples:

$\mathbf{LJ} + (p \rightarrow q) \vee (q \rightarrow p)$	Gödel logic
$\mathbf{LJ} + (\neg p \vee \neg\neg p)$	Jankov logic
$\mathbf{LJ} + p_0 \vee (p_0 \rightarrow p_1) \vee \ldots (p_0 \wedge \ldots \wedge p_{k-1} \rightarrow p_n)$	Bwk

- Showed that

$$\mathbf{LJ} + (p \to q \vee r) \vee (q \to p \vee r) \vee (r \to p \vee q) \quad \text{Bc3}$$

is not variable-bounded.

A New Proof of an Old Result

S5 = modal logic of *reflexive, transitive and symmetric* Kripke frames

- Ohnishi and Matsumoto (1957):

$$s\mathbf{S5} \quad = \quad \mathbf{LK} \quad + \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\; T \quad + \quad \frac{\Box\Gamma \Rightarrow A, \Box\Delta}{\Box\Gamma \Rightarrow \Box A, \Box\Delta}\; 5$$

- ______ (1959): cut elimination fails in s**S5**
- Pottinger (1983), Avron (1996), Kurokawa (2013), . . . cut-free hypersequent systems for **S5**

- Takano (1992): *s***S5** has the analytic cut property

$$\frac{\Gamma \Rightarrow C, \Delta \quad \Pi, C \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ \textit{cut}$$

  $C \in \mathrm{Subformula}(\Gamma, \Pi \Rightarrow \Delta, \Sigma)$.
- Proof by (rather intricate) proof transformations in *s***S5**.
- Can we get Takano's result by our method?

Kurokawa's hypersequent calculus $h\mathbf{S5}$:

$$h\mathbf{LK} \quad + \quad \frac{G \mid \Gamma, A \Rightarrow \Delta}{G \mid \Gamma, \Box A \Rightarrow \Delta}\, T \quad + \quad \frac{G \mid \Box\Gamma \Rightarrow A}{G \mid \Box\Gamma \Rightarrow \Box A}\, 4$$

$$+ \quad \frac{G \mid \Box C, \Pi \Rightarrow \Sigma}{G \mid \Box C \Rightarrow \mid \Pi \Rightarrow \Sigma}\, MS$$

Theorem (Kurokawa 2013)

$h\mathbf{S5}$ is cut-free complete for $\mathbf{S5}$.

$$\cfrac{\overset{\delta}{G \mid \Box C, \Pi \Rightarrow \Sigma}}{G \mid \Box C \Rightarrow \mid \Pi \Rightarrow \Sigma}\ MS$$

$$\vdots$$

$$\Rightarrow F$$

LEFT SPLIT:

$$\cfrac{\Box C \Rightarrow \Box C}{G \mid \Box C \Rightarrow \boxed{\Box C} \mid \Pi \Rightarrow \Sigma}\ ew$$

$$\vdots$$

$$\Rightarrow F, \boxed{\Box C}$$

$$\dfrac{\overset{\delta}{G \mid \Box C, \Pi \Rightarrow \Sigma}}{G \mid \Box C \Rightarrow \mid \Pi \Rightarrow \Sigma}\ MS$$

$$\vdots$$

$$\Rightarrow F$$

RIGHT SPLIT:

$$\dfrac{\overset{\delta}{\boxed{\Box C}, \Pi \Rightarrow \Sigma}}{G \mid \Box C \Rightarrow \mid \boxed{\Box C}, \Pi \Rightarrow \Sigma}\ ew$$

$$\vdots$$

$$\boxed{\Box C} \Rightarrow F$$

$$\dfrac{\overset{\text{LEFT SPLIT}}{\vdots}\atop \Rightarrow F, \boxed{\Box C} \qquad \overset{\text{RIGHT SPLIT}}{\vdots}\atop \boxed{\Box C} \Rightarrow F}{\Rightarrow F}\ cut$$

$\Box C$ is a subformula of F

$\Rightarrow$ the cut is analytic

Theorem (Takano 1992, new proof)

*s***S5** *has the analytic cut property.*

Recapitulation & Questions

1. For many extensions $\mathbf{FL_{e(wc)}} + A$,

$\mathbf{FL_{e(wc)}} + A$ has a cut-free hypersequent calculus	$\Rightarrow$	$\mathbf{FL_{e(wc)}} + A$ is multiset-bounded

2. set-boundedness allows reduction to base logic $\mathbf{FL_{e(wc)}}$ ($\rightarrow$ decidability & complexity upper bound)

- ? Can we prove other metalogical properties of $\mathbf{FL_{e(wc)}} + A$ from (multi)set-boundedness? (i.e., interpolation?)
- ? cut-free hypersequents $\overset{\text{proof complexity?}}{\longleftrightarrow}$ bounded sequent proofs
- ? Find out more on the relation between (multiset/set/formula/variable)-boundedness

- ? Can we prove other metalogical properties of $\mathbf{FL_{e(wc)}} + A$ from (multi)set-boundedness? (i.e., interpolation?)
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Thank your for listening!