Nested Sequents and Countermodels for Monotone Modal Logic

Björn Lellmann

TICAMORE workshop 2019, Vienna
Nov 11-23, 2019

Modal logics: A success story

Fact

Many problems in Computer Science are modelled in Modal Logic.

Examples

- Epistemic logics: $\mathcal{K}(A)$ ... "the agent knows A is the case"
- Deontic logics: $\mathcal{O}(A)$ ... "A ought to be the case"
- ...

In particular, modal logics often have nice reasoning systems a.k.a. calculi with strong connections to

- Syntax: useful for proving theorems
- Semantics: useful for finding countermodels.

Modal logics: A success story (normally?)

...But not all applications might satisfy normality:

Epistemic logics: $\mathcal{K}(A)$ ... "the agent knows that A is the case"

- $\mathcal{K}(\top)$ ... "the agent knows all tautologies"

Deontic logics: $\mathcal{O}(A)$ ... "A ought to be the case"

- $\mathcal{O}(\mathrm{go}) \wedge \mathcal{O}(\neg\mathrm{go}) \to \mathcal{O}(\mathrm{go} \wedge \neg\mathrm{go})$ ... "in presence of conflicting obligations, $\perp$ ought to be the case"

So...

Can we find good calculi for non-normal modal logics?

Monotone modal logic

The formulae of monotone modal logic M are given by

$$p \in \mathsf{Var} \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \langle]\varphi$$

A neighbourhood frame $\mathcal{F} = (W, \mathcal{N})$ has a neighbourhood function satisfying $\mathcal{N}(w) \subseteq \mathcal{P}(W)$ for every $w \in W$.
Valuations $[\![.]\!]$ satisfy:

- local clauses for $\wedge, \vee, \to, \bot$.
- $\mathcal{F}, [\![.]\!], w \Vdash \langle]A$ iff $\exists \alpha \in \mathcal{N}(w)\, \forall v \in \alpha.\, \mathcal{F}, [\![.]\!], v \Vdash A$

The axiomatisation of M is given by propositional logic and the rule

$$\frac{\vdash A \to B}{\vdash \langle]A \to \langle]B}$$

Reasoning in monotone modal logic

There are some calculi for M:

Syntactical calculi:

- Sequent calculi [Lavendhomme, Lucas:2000, Indrzejczak:2005]
- (Labelled Tableaux [Indrzejczak:2007])

Pro: Good for reasoning, formula interpretation

Con: Bad for countermodels

Semantical calculi:

- Labelled sequent calculi [Negri:2017, Dalmonte et al:2018]

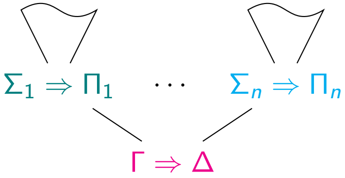
Pro: Good for countermodels

Con: Bad for reasoning, no formula interpretation

Can we get the best of both worlds?

Nested sequents to the rescue!

Nested sequents are trees of (multi-set based) sequents:



interpreted in normal modal logics as

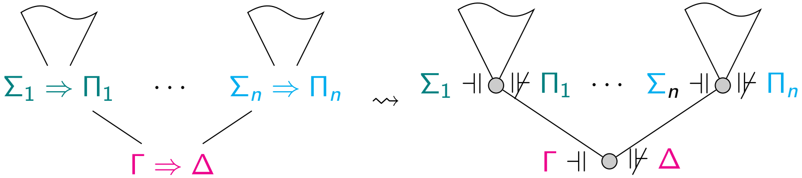
$$\bigwedge \Gamma \to \bigvee \Delta \vee \Box(\bigwedge \Sigma_1 \to \bigvee \Pi_1^*) \vee \cdots \vee \Box(\bigwedge \Sigma_n \to \bigvee \Pi_n^*).$$

A bit of history:

- Precursors: [Bull:'92], [Kashima:'94], [Masini:'92]
- Current form in modal logics: [Brünnler:'09], [Poggiolesi:'09]
- For intuitionistic modal logics: [Straßburger et al:'12 - now]
- Adapted to intuitionistic logic in [Fitting:'14]

Nested sequents to the rescue!

Nested sequents are trees of (multi-set based) sequents:



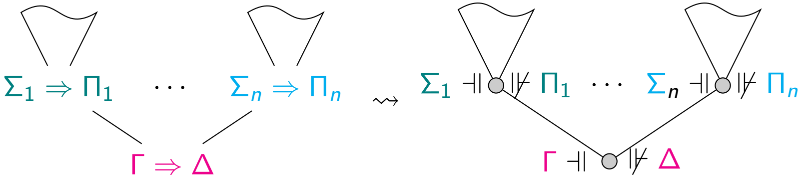
interpreted in normal modal logics as

$$\bigwedge\Gamma \to \bigvee\Delta \vee \Box(\bigwedge\Sigma_1 \to \bigvee\Pi_1^*) \vee \cdots \vee \Box(\bigwedge\Sigma_n \to \bigvee\Pi_n^*).$$

Nested sequents give rise to models for normal modal logic.

Nested sequents to the rescue!

Nested sequents are trees of (multi-set based) sequents:



interpreted in normal modal logics as

$$\bigwedge\Gamma \to \bigvee\Delta \vee \Box(\bigwedge\Sigma_1 \to \bigvee\Pi_1^*) \vee \cdots \vee \Box(\bigwedge\Sigma_n \to \bigvee\Pi_n^*).$$

Nested sequents give rise to models for normal modal logic.

But:

- How to construct countermodels for non-normal logics?
- "Deep applicability" of the rules means we cannot have a formula interpretation in the non-normal language.

What's the problem with the formula interpretation?

Interpreting the nesting of nested sequents with τ and using Ackermann's Lemma we have the following equivalences:

$$\frac{\Gamma \Rightarrow \Delta, [\, \Rightarrow A] \quad \Gamma \Rightarrow \Delta, [\, \Rightarrow B]}{\Gamma \Rightarrow \Delta, [\, \Rightarrow A \wedge B]} \quad \leftrightsquigarrow \quad \overline{\tau(A) \wedge \tau(B) \Rightarrow \tau(A \wedge B)}$$

$$\overline{\Rightarrow [p \Rightarrow p]} \quad \leftrightsquigarrow \quad \overline{\Rightarrow \tau(p \to p)}$$

$$\frac{\Gamma \Rightarrow \Delta, [\, \Rightarrow A]}{\Gamma \Rightarrow \Delta, [\, \Rightarrow A \vee B]} \quad \leftrightsquigarrow \quad \overline{\tau(A) \Rightarrow \tau(A \vee B)}$$

Note that these are (equivalent to) the axioms of K. Hence:

"Deep applicability" of the propositional rules implies normality of the interpretation of the nesting operator!

monotone modal logic

The formulae of monotone modal logic are given by

$$p \in \mathsf{Var} \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \langle]\varphi$$

A neighbourhood frame $\mathcal{F} = (W, \mathcal{N})$ has a neighbourhood function satisfying $\mathcal{N}(w) \subseteq \mathcal{P}(W)$ for every $w \in W$.
Valuations $[\![.]\!]$ satisfy:

- local clauses for $\wedge, \vee, \to, \bot$.
- $\mathcal{F}, [\![.]\!], w \Vdash \langle]A$ iff $\exists \alpha \in \mathcal{N}(w)\, \forall v \in \alpha.\ \mathcal{F}, [\![.]\!], v \Vdash A$

Bimodal monotone modal logic

The formulae of bimodal monotone modal logic aka. Brown's Ability Logic are given by

$$p \in \mathsf{Var} \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \langle]\varphi \mid []\varphi$$

A neighbourhood frame $\mathcal{F} = (W, \mathcal{N})$ has a neighbourhood function satisfying $\mathcal{N}(w) \subseteq \mathcal{P}(W)$ for every $w \in W$.
Valuations $[\![.]\!]$ satisfy:

- local clauses for $\wedge, \vee, \to, \bot$.
- $\mathcal{F}, [\![.]\!], w \Vdash \langle]A$ iff $\exists \alpha \in \mathcal{N}(w)\, \forall v \in \alpha.\, \mathcal{F}, [\![.]\!], v \Vdash A$
- $\mathcal{F}, [\![.]\!], w \Vdash []A$ iff $\forall \alpha \in \mathcal{N}(w)\, \forall v \in \alpha.\, \mathcal{F}, [\![.]\!], v \Vdash A$

Brown's ability interpretation [Brown:'88]:

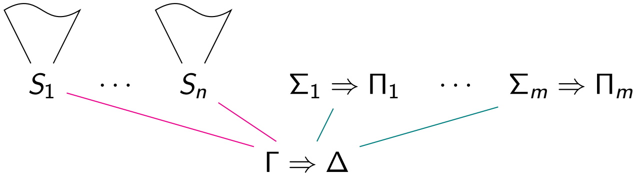
$\langle]A$: "The agent can reliably bring about A"
$[]A$: "The agent will bring about A"

Bimodal nested sequents

A bimodal nested sequent is a structure

$$\Gamma \Rightarrow \Delta, [S_1], \ldots, [S_n], \langle \Sigma_1 \Rightarrow \Pi_1 \rangle, \ldots, \langle \Sigma_m \Rightarrow \Pi_m \rangle$$

with $n, m \geq 0$ where the S_i are bimodal nested sequents.
As a tree:



Its formula interpretation ι is

$$\bigwedge\Gamma \to \bigvee\Delta \ \vee \ \bigvee_{i=1}^{n} [\,]\iota(S_i) \ \vee \ \bigvee_{j=1}^{m} \langle\,]\left(\bigwedge\Sigma_j \to \bigvee\Pi_j\right)$$

The calculus for bimodal M

The calculus contains the (classical) propositional rules plus:

$$\frac{\Gamma \Rightarrow \Delta, [\ \Rightarrow A]}{\Gamma \Rightarrow \Delta, [\,]A}\ [\,]_R \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]}{\Gamma, [\,]A \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]}\ [\,]_L$$

$$\frac{\Gamma \Rightarrow \Delta, \langle\ \Rightarrow A\rangle}{\Gamma \Rightarrow \Delta, \langle]A}\ \langle]_R \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]}{\Gamma, \langle]A \Rightarrow \Delta, \langle\Sigma \Rightarrow \Pi\rangle}\ \langle]_L$$

$$\frac{\Gamma \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]}{\Gamma \Rightarrow \Delta, [\,]A, \langle\Sigma \Rightarrow \Pi\rangle}\ \mathsf{I}$$

Rules are applied anywhere except inside $\langle . \rangle$.

Theorem

The rules are sound wrt. the formula interpretation and (a variant of) the calculus has cut elimination.

The calculus for bimodal M

The calculus contains the (classical) propositional rules plus:

$$\frac{\Gamma \Rightarrow \Delta, [\Rightarrow A]}{\Gamma \Rightarrow \Delta, [\,]A}\ [\,]_R \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]}{\Gamma, [\,]A \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]}\ [\,]_L$$

$$\frac{\Gamma \Rightarrow \Delta, \langle \Rightarrow A\rangle}{\Gamma \Rightarrow \Delta, \langle]A}\ \langle]_R \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]}{\Gamma, \langle]A \Rightarrow \Delta, \langle\Sigma \Rightarrow \Pi\rangle}\ \langle]_L$$

$$\frac{\Gamma \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]}{\Gamma \Rightarrow \Delta, [\,]A, \langle\Sigma \Rightarrow \Pi\rangle}\ \mathsf{I}$$

Rules are applied anywhere except inside $\langle . \rangle$.

Bonus: Restricting the language specifies the calculus to the standard (linear) nested sequent calculus for modal logic K

The calculus for bimodal M

The calculus contains the (classical) propositional rules plus:

$$\frac{\Gamma \Rightarrow \Delta, [\ \Rightarrow A]}{\Gamma \Rightarrow \Delta, [\,]A}\ [\,]_R \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]}{\Gamma, [\,]A \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]}\ [\,]_L$$

$$\frac{\Gamma \Rightarrow \Delta, \langle\ \Rightarrow A\rangle}{\Gamma \Rightarrow \Delta, \langle\,]A}\ \langle\,]_R \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]}{\Gamma, \langle\,]A \Rightarrow \Delta, \langle\Sigma \Rightarrow \Pi\rangle}\ \langle\,]_L$$

$$\frac{\Gamma \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]}{\Gamma \Rightarrow \Delta, [\,]A, \langle\Sigma \Rightarrow \Pi\rangle}\ \mathsf{I}$$

Rules are applied anywhere except inside $\langle . \rangle$.

Bonus: Restricting the language specifies the calculus to the standard (linear) nested sequent calculus for modal logic K or the (linear) nested sequent calculus for monomodal M

Deontic extensions

Adding further rules gives calculi for extensions of the logic. Some (vaguely) deontic ones:

	axiom	frame property	$P_{\langle]}$	$N_{\langle]}$	$D_{[]}$
$\mathsf{n}_{\langle]}$	$\neg\langle]\bot$	$\emptyset \notin \mathcal{N}(w)$	✓		
$\mathsf{d}_{\langle]}$	$\neg(\langle]A \wedge [\,]\neg A)$	$\emptyset \notin \mathcal{N}(w)$	✓		
$\mathsf{d}_{[\rangle}$	$[\,]A \to \langle]A$	$\mathcal{N}(w) \neq \emptyset$		✓	
$\mathsf{d}_{[]}$	$\neg([\,]A \wedge [\,]\neg A)$	$\exists\alpha \in \mathcal{N}(w).\alpha \neq \emptyset$		✓	✓

With the additional rules:

$$\frac{\Gamma \Rightarrow \Delta, \langle \Rightarrow \rangle}{\Gamma \Rightarrow \Delta}\ P_{\langle]} \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]}{\Gamma \Rightarrow \Delta, \langle\Sigma \Rightarrow \Pi\rangle}\ N_{\langle]} \qquad \frac{\Gamma \Rightarrow \Delta, [\Rightarrow]}{\Gamma \Rightarrow \Delta}\ D_{[]}$$

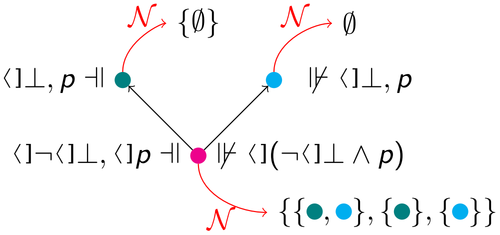
Bonus: Restricting the language specifies the calculus to those for MP (via $\mathsf{n}_{\langle]}$) and for KD (via $\mathsf{d}_{[]}$).

What about countermodels?

Using an annotated version of the calculus, underivable sequents give rise to countermodels: E.g.

$$\langle]\neg\langle]\bot, \langle]p \Rightarrow \langle](\neg\langle]\bot \wedge \langle]p),\ [\,\langle]\bot, p \Rightarrow\,],\ [\,\Rightarrow \langle]\bot, p\,]$$

yields



Theorem

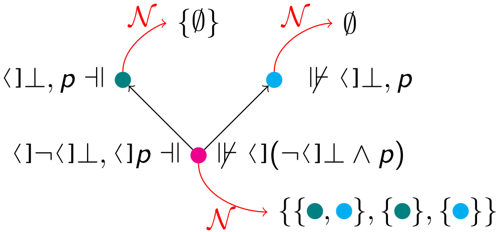
The calculus for bimodal M *is cut-free complete and failed proof search yields a countermodel.*

What about countermodels?

Using an annotated version of the calculus, underivable sequents give rise to countermodels: E.g.

$$\langle]\neg\langle]\bot, \langle]p \Rightarrow \langle](\neg\langle]\bot \wedge \langle]p),\ [\,\langle]\bot, p \Rightarrow\,],\ [\,\Rightarrow \langle]\bot, p\,]$$

yields



Corollary (Bonus)

The calculi for K *and monomodal* M *are cut-free complete and failed proof search yields a countermodel.*

What do derivations look like?

. . . Let the implementation work that out!
(`http://subsell.logic.at/bprover/nnProver/`)

Suming up

Bimodal nested sequents for monotone modal logic yield:

- an internal calculus;
- syntactic cut elimination;
- support for countermodel construction;
- the basis for a general treatment of non-normal modal logics;
- an implementation including countermodel generation

Thank You!