# On Deriving Nested Calculi for Intuitionistic Logics from Semantic Systems

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Translating and Discovering Calculi for Modal and Related Logics (TICAMORE WORKSHOP V 2019)

# Motivation

Labelled (Semantic) Paradigm:

- Semantic Clauses + Frame Properties $\rightarrow$ Inference Rules
- Easy to Construct (Process can be automated)
- General Theorems:
  - Cut-Elimination
  - Invertibility of Rules
  - Structural Rule Admissibility

# Motivation

Nested (Syntactic) Paradigm:

- Minimize Bureaucracy
- Strong Subformula Property (Useful for Automated Reasoning)

*Extracting latter from former (preserving properties) would be useful.

*Case Study: Intuitionistic Logics

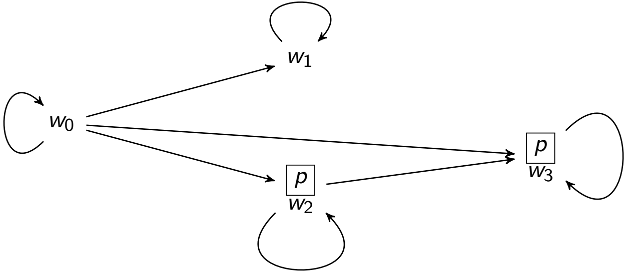
# Overview of Talk

1. Intuitionistic Logics
2. Labelled Sequents and Calculi
3. Nested Sequents and Calculi
4. Extracting Nested from Labelled
5. Future Work

# Propositional Intuitionistic Logic

Language: $A ::= p \mid \bot \mid (A \vee A) \mid (A \wedge A) \mid (A \rightarrow A)$

Propositional Intuitionistic Model $(W, \leq, V)$

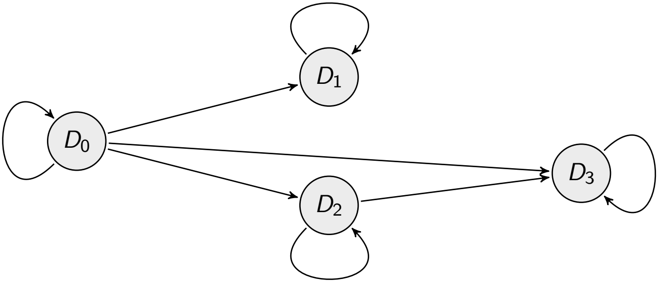


# First-Order Intuitionistic Logic (Constant Domains)

Language:

$$A ::= p(x_1, \ldots, x_n) \mid \bot \mid (A \vee A) \mid (A \wedge A) \mid (A \rightarrow A) \mid (\exists x)A \mid (\forall x)A$$

First-Order Intuitionistic Model $(W, \leq, D, V)$



# Labelled Sequents

Labelled Sequents: $L_1 \Rightarrow L_2$ where

$$L_1 ::= w : A \mid a \in D_w \mid w \leq v \mid L_1, L_1 \qquad L_2 ::= w : A \mid L_2, L_2$$

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Example:

$$w \leq w, w \leq u, u \leq v, w \leq v, w \leq z, w : A, z : D \Rightarrow w : B, u : C$$

$$\begin{tikzcd} \underset{w}{\boxed{A \Rightarrow B}} \arrow[loop above] \arrow[r] \arrow[d] \arrow[dr] & \underset{z}{\boxed{D \Rightarrow \emptyset}} \\ \underset{u}{\boxed{\emptyset \Rightarrow C}} \arrow[r] & \underset{v}{\boxed{\emptyset \Rightarrow \emptyset}} \end{tikzcd}$$

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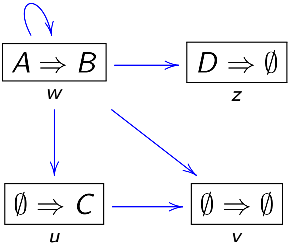

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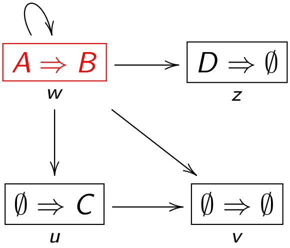

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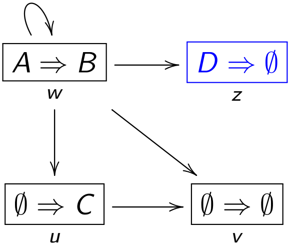

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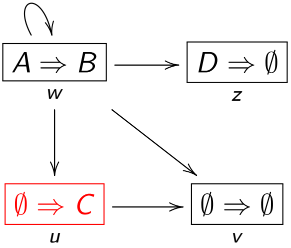

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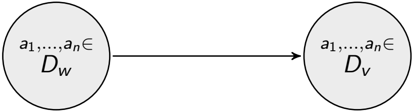
\end{tikzcd}$$

# Labelled Calculus (IntQC): Initial Sequents

$$\frac{}{\mathcal{R}, w \leq v, \vec{a} \in D_w, w : p(\vec{a}), \Gamma \Rightarrow \Delta, v : p(\vec{a})} (id_q)$$




Notation:

- $\vec{a} = (a_1, \ldots, a_n)$
- $\vec{a} \in D_w = a_1 \in D_w, \ldots, a_n \in D_w$

# Labelled Calculus (IntQC): Propositional Connective Rules

$$\dfrac{}{\mathcal{R}, w : \bot, \Gamma \Rightarrow \Delta}\ (\bot_l) \qquad \dfrac{\mathcal{R}, w \leq v, v : A, \Gamma \Rightarrow \Delta, v : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, w : A \supset B}\ (\supset_r)^{\dagger_1}$$

$$\dfrac{\mathcal{R}, w : A, w : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, w : A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge_l) \qquad \dfrac{\mathcal{R}, \Gamma \Rightarrow \Delta, w : A \qquad \mathcal{R}, \Gamma \Rightarrow \Delta, w : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, w : A \wedge B}\ (\wedge_r)$$

$$\dfrac{\mathcal{R}, w : A, \Gamma \Rightarrow \Delta \qquad \mathcal{R}, w : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, w : A \vee B, \Gamma \Rightarrow \Delta}\ (\vee_l) \qquad \dfrac{\mathcal{R}, \Gamma \Rightarrow \Delta, w : A, w : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, w : A \vee B}\ (\vee_r)$$

$$\dfrac{\mathcal{R}, w \leq v, w : A \supset B, \Gamma \Rightarrow \Delta, v : A \qquad \mathcal{R}, w \leq v, w : A \supset B, v : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, w \leq v, w : A \supset B, \Gamma \Rightarrow \Delta}\ (\supset_l)$$

# Labelled Calculus (IntQC): Structural Rules

$$\frac{\mathcal{R}, w \leq w, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta}\,(ref) \quad \frac{\mathcal{R}, w \leq v, v \leq u, w \leq u, \Gamma \Rightarrow \Delta}{\mathcal{R}, w \leq v, v \leq u, \Gamma \Rightarrow \Delta}\,(tra)$$

$$\frac{\mathcal{R}, w \leq v, a \in D_w, a \in D_v, \Gamma \Rightarrow \Delta}{\mathcal{R}, w \leq v, a \in D_w, \Gamma \Rightarrow \Delta}\,(nd)$$

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# Labelled Calculus (IntQC): Quantifier Rules

$$\frac{\mathcal{R}, w \leq v, a \in D_v, \Gamma \Rightarrow \Delta, v : A[a/x]}{\mathcal{R}, \Gamma \Rightarrow \Delta, w : \forall x A} \ (\forall_r)^{\dagger_2}$$

$$\frac{\mathcal{R}, a \in D_w, \Gamma \Rightarrow \Delta, w : A[a/x], w : \exists x A}{\mathcal{R}, a \in D_w, \Gamma \Rightarrow \Delta, w : \exists x A} \ (\exists_r)$$

$$\frac{\mathcal{R}, a \in D_w, w : A[a/x], \Gamma \Rightarrow \Delta}{\mathcal{R}, w : \exists x A, \Gamma \Rightarrow \Delta} \ (\exists_l)^{\dagger_3}$$

$$\frac{\mathcal{R}, w \leq v, a \in D_v, v : A[a/x], w : \forall x A, \Gamma \Rightarrow \Delta}{\mathcal{R}, w \leq v, a \in D_v, w : \forall x A, \Gamma \Rightarrow \Delta} \ (\forall_l)$$

# Properties of Labelled Calculus G3IntQC

1. Sound and Complete
2. Hp-admissibility of substitutions
3. Hp-invertibility of rules
4. Hp-admissibility of structural rules
5. Cut-Elimination

# Nested Sequents

Language:

$$X ::= A \mid X, X \qquad \Sigma ::= X \to X \mid X \to X, [\Sigma], \ldots, [\Sigma]$$

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Example: $X_0 \to Y_0, [X_1 \to Y_1, [X_2 \to Y_2]], [X_3 \to Y_3]$

$$\begin{tikzcd}
\underset{0}{\boxed{X_0 \to Y_0}} \arrow[r] \arrow[d] & \underset{01}{\boxed{X_3 \to Y_3}} \\
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\end{tikzcd}$$

# Nested Calculus for IntQC: Propositional Fragment

$$\frac{}{\Sigma[X, p(\vec{a}) \to p(\vec{a}), Y]}\ (id_q)$$

$$\frac{\Sigma[X, A, B \to Y]}{\Sigma[X, A \wedge B \to Y]}\ (\wedge_l) \qquad \frac{\Sigma[X \to A, Y] \qquad \Sigma[X \to B, Y]}{\Sigma[X \to A \wedge B, Y]}\ (\wedge_r)$$

$$\frac{\Sigma[X, A \to Y] \qquad \Sigma[X, B \to Y]}{\Sigma[X, A \vee B \to Y]}\ (\vee_l) \qquad \frac{\Sigma[X \to A, B, Y]}{\Sigma[X \to A \vee B, Y]}\ (\vee_r)$$

$$\frac{\Sigma[X \to Y, [A \to]]}{\Sigma[X \to Y, \neg A]}\ (\neg_r) \qquad \frac{\Sigma[X \to A, Y]}{\Sigma[X, \neg A \to Y]}\ (\neg_l)$$

$$\frac{\Sigma[X \to Y, [A \to B]]}{\Sigma[X \to A \supset B, Y]}\ (\supset_r) \qquad \frac{\Sigma[X \to A, Y] \qquad \Sigma[X, B \to Y]}{\Sigma[X, A \supset B \to Y]}\ (\supset_l)$$

$$\frac{\Sigma\{X \to Y, [X', A \to Y']\}}{\Sigma\{X, A \to Y, [X' \to Y']\}}\ (lift)$$

# Nested Calculus for IntQC: Quantifier Rules

$$\frac{\Sigma[X \to Y, A[a/x]]}{\Sigma[X \to Y, \exists x A]} (\exists_r) \qquad \frac{\Sigma[X \to Y, A[a/x]]}{\Sigma[X \to Y, \forall x A]} (\forall_r)^{\dagger}$$

$$\frac{\Sigma[X, A[a/x] \to Y]}{\Sigma[X, \forall x A \to Y]} (\forall_l) \qquad \frac{\Sigma[X, A[a/x] \to Y]}{\Sigma[X, \exists x A \to Y]} (\exists_l)^{\dagger}$$

# Properties of Nested Calculus NIntQC

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By Fitting's 2014 paper we have:

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*Through our extraction method, we will obtain more properties!

# Eliminating Reflexivity

Two Issues:

$$\frac{}{\mathcal{R}, w \leq v, \vec{a} \in D_w, w : p(\vec{a}), \Gamma \Rightarrow \Delta, v : p(\vec{a})} \ (id_q)$$

$$\frac{\mathcal{R}, w \leq v, w : A \supset B, \Gamma \Rightarrow \Delta, v : A \qquad \mathcal{R}, w \leq v, w : A \supset B, v : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, w \leq v, w : A \supset B, \Gamma \Rightarrow \Delta} \ (\supset_l)$$

# Eliminating Reflexivity: $(id_q)$ Case

$$\dfrac{\overline{\mathcal{R}, w \leq w, \vec{a} \in D_w, w : p(\vec{a}), \Gamma \Rightarrow \Delta, w : p(\vec{a})}\ (id_q)}{\mathcal{R}, \vec{a} \in D_w, w : p(\vec{a}), \Gamma \Rightarrow \Delta, w : p(\vec{a})}\ (ref)$$

$$\underset{w}{\boxed{p(\vec{a}) \Rightarrow p(\vec{a})}} \quad \Big| \quad \underset{w}{\boxed{p(\vec{a}) \Rightarrow p(\vec{a})}}$$

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$$\underset{w}{\boxed{p(\vec{a}) \Rightarrow p(\vec{a})}} \qquad \Bigg| \qquad \underset{w}{\boxed{p(\vec{a}) \Rightarrow p(\vec{a})}}$$

- Problem: Conclusion is not an initial sequent

# Eliminating Reflexivity: $(id_q)$ Case

$$\dfrac{\dfrac{}{\mathcal{R}, w \leq w, \vec{a} \in D_w, w : p(\vec{a}), \Gamma \Rightarrow \Delta, w : p(\vec{a})}\ (id_q)}{\mathcal{R}, \vec{a} \in D_w, w : p(\vec{a}), \Gamma \Rightarrow \Delta, w : p(\vec{a})}\ (ref)$$

$$\underset{w}{\boxed{p(\vec{a}) \Rightarrow p(\vec{a})}} \quad \Bigg| \quad \underset{w}{\boxed{p(\vec{a}) \Rightarrow p(\vec{a})}}$$

- Problem: Conclusion is not an initial sequent
- Solution: Add the following as an initial sequent:

$$\dfrac{}{\mathcal{R}, \vec{a} \in D_w, w : p(\vec{a}), \Gamma \Rightarrow \Delta, w : p(\vec{a})}\ (id_q^*)$$

# Eliminating Reflexivity: $(\supset_l)$ Case

$$\dfrac{\dfrac{\mathcal{R}, w \leq w, w : A \supset B, \Gamma \Rightarrow \Delta, w : A \qquad \mathcal{R}, w \leq w, w : A \supset B, w : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, w \leq w, w : A \supset B, \Gamma \Rightarrow \Delta}\,(\supset_l)}{\mathcal{R}, w : A \supset B, \Gamma \Rightarrow \Delta}\,(ref)$$

# Eliminating Reflexivity: $(\supset_l)$ Case

$$\dfrac{\dfrac{\mathcal{R}, w \leq w, w : A \supset B, \Gamma \Rightarrow \Delta, w : A \qquad \mathcal{R}, w \leq w, w : A \supset B, w : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, w \leq w, w : A \supset B, \Gamma \Rightarrow \Delta}\,(\supset_l)}{\mathcal{R}, w : A \supset B, \Gamma \Rightarrow \Delta}\,(ref)$$

- Let's try to permute $(ref)$ upwards:

$$\dfrac{\mathcal{R}, {\color{red}w \leq w, w : A \supset B}, \Gamma \Rightarrow \Delta, {\color{red}w : A}}{\mathcal{R}, {\color{red}w : A \supset B}, \Gamma \Rightarrow \Delta, {\color{red}w : A}}\,(ref) \qquad \dfrac{\mathcal{R}, {\color{blue}w \leq w, w : A \supset B, w : B}, \Gamma \Rightarrow \Delta}{\mathcal{R}, {\color{blue}w : A \supset B, w : B}, \Gamma \Rightarrow \Delta}\,(ref)$$

# Eliminating Reflexivity: $(\supset_l)$ Case

$$\dfrac{\dfrac{\mathcal{R}, w \leq w, w : A \supset B, \Gamma \Rightarrow \Delta, w : A \qquad \mathcal{R}, w \leq w, w : A \supset B, w : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, w \leq w, w : A \supset B, \Gamma \Rightarrow \Delta}(\supset_l)}{\mathcal{R}, w : A \supset B, \Gamma \Rightarrow \Delta}(ref)$$

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- Solution: Add the following rule:

$$\dfrac{\mathcal{R}, w : A \supset B, \Gamma \Rightarrow \Delta, w : A \qquad \mathcal{R}, w : A \supset B, w : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, w : A \supset B, \Gamma \Rightarrow \Delta}(\supset_l^*)$$

# Eliminating Transitivity

$$
\dfrac{\dfrac{}{\mathcal{R}, w \leq u, u \leq v, w \leq v, \vec{a} \in D_w, w : p(\vec{a}), \Gamma \Rightarrow \Delta, v : p(\vec{a})}\ (id_q)}{\mathcal{R}, w \leq u, u \leq v, \vec{a} \in D_w, w : p(\vec{a}), \Gamma \Rightarrow \Delta, v : p(\vec{a})}\ (tra)
$$

$$
\begin{tikzcd}
\underset{w}{\boxed{p(\vec{a}) \Rightarrow \emptyset}} \arrow[d] \arrow[dr] & \\
\underset{u}{\boxed{\emptyset \Rightarrow \emptyset}} \arrow[r] & \underset{v}{\boxed{\emptyset \Rightarrow p(\vec{a})}}
\end{tikzcd}
\qquad\Bigg|\qquad
\begin{tikzcd}
\underset{w}{\boxed{p(\vec{a}) \Rightarrow \emptyset}} \arrow[d] & \\
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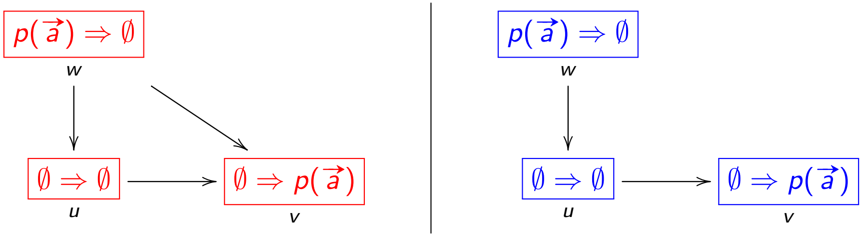
\end{tikzcd}
$$

# Eliminating Transitivity

$$\dfrac{\dfrac{}{\mathcal{R}, w \leq u, u \leq v, w \leq v, \vec{a} \in D_w, w : p(\vec{a}), \Gamma \Rightarrow \Delta, v : p(\vec{a})}\,(id_q)}{\mathcal{R}, w \leq u, u \leq v, \vec{a} \in D_w, w : p(\vec{a}), \Gamma \Rightarrow \Delta, v : p(\vec{a})}\,(tra)$$




- Problem: Conclusion is not an initial sequent

# Eliminating Transitivity

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\end{tikzcd}
$$

- Problem: Conclusion is not an initial sequent
- Solution:

$$
\dfrac{\mathcal{R}, w \leq v, w : A, v : A, \Gamma \Rightarrow \Delta}{\mathcal{R}, w \leq v, w : A, \Gamma \Rightarrow \Delta}\,(lift)
$$

# Eliminating Transitivity

$$\dfrac{\dfrac{}{\mathcal{R}, w \leq u, u \leq v, \vec{a} \in D_w, w : p(\vec{a}), u : p(\vec{a}), \Gamma \Rightarrow \Delta, v : p(\vec{a})}\,(id_q)}{\mathcal{R}, w \leq u, u \leq v, \vec{a} \in D_w, w : p(\vec{a}), \Gamma \Rightarrow \Delta, v : p(\vec{a})}\,(lift)$$

$$\begin{tikzcd} \underset{w}{\boxed{p(\vec{a}) \Rightarrow \emptyset}} \arrow[d] & \\ \underset{u}{\boxed{p(\vec{a}) \Rightarrow \emptyset}} \arrow[r] & \underset{v}{\boxed{\emptyset \Rightarrow p(\vec{a})}} \end{tikzcd} \qquad \begin{tikzcd} \underset{w}{\boxed{p(\vec{a}) \Rightarrow \emptyset}} \arrow[d] & \\ \underset{u}{\boxed{\emptyset \Rightarrow \emptyset}} \arrow[r] & \underset{v}{\boxed{\emptyset \Rightarrow p(\vec{a})}} \end{tikzcd}$$

# Consequences of (*ref*) and (*tra*) Elimination

**Theorem**

*Every derivation in* G3IntQC - $\{(\mathit{ref}), (\mathit{tra})\} + \{(\mathit{id}_q^*), (\supset_l^*), (\mathit{lift})\}$ *of a labelled theorem* $w : A$ *is treelike.*

Treelike = (1) Connected and (2) Acyclic

# Consequences of $(ref)$ and $(tra)$ Elimination II

$$\dfrac{\dfrac{\dfrac{}{w \leq u, u \leq u, u : p(\vec{a}) \Rightarrow u : p(\vec{a})}\,(id_q)}{w \leq u, u : p(\vec{a}) \Rightarrow u : p(\vec{a})}\,(ref)}{\Rightarrow w : p(\vec{a}) \supset p(\vec{a})}\,(\supset_r)$$

$$\begin{tikzcd} \underset{w}{\boxed{\emptyset \Rightarrow \emptyset}} \arrow[loop above] \arrow[r] & \underset{w}{\boxed{p(\vec{a}) \Rightarrow p(\vec{a})}} \end{tikzcd}$$

$$\begin{tikzcd} \underset{w}{\boxed{\emptyset \Rightarrow \emptyset}} \arrow[r] & \underset{w}{\boxed{p(\vec{a}) \Rightarrow p(\vec{a})}} \end{tikzcd}$$

$$\underset{w}{\boxed{\emptyset \Rightarrow p(\vec{a}) \supset p(\vec{a})}}$$

# Consequences of (*ref*) and (*tra*) Elimination III

$$\dfrac{\dfrac{}{w \leq u, u : p(\vec{a}) \Rightarrow u : p(\vec{a})}\ (id_q^*)}{\Rightarrow w : p(\vec{a}) \supset p(\vec{a})}\ (\supset_r)$$

$$\underset{w}{\boxed{\emptyset \Rightarrow \emptyset}} \longrightarrow \underset{w}{\boxed{p(\vec{a}) \Rightarrow p(\vec{a})}}$$

$$\underset{w}{\boxed{\emptyset \Rightarrow p(\vec{a}) \supset p(\vec{a})}}$$

# Eliminating Domain Conditions

We want to eliminate the following:

$$\frac{\mathcal{R}, w \leq v, a \in D_w, a \in D_v, \Gamma \Rightarrow \Delta}{\mathcal{R}, w \leq v, a \in D_w, \Gamma \Rightarrow \Delta}\,(nd) \qquad \frac{\mathcal{R}, w \leq v, a \in D_v, a \in D_w, \Gamma \Rightarrow \Delta}{\mathcal{R}, w \leq v, a \in D_v, \Gamma \Rightarrow \Delta}\,(cd)$$

Main Issues:

$$\frac{}{\mathcal{R}, \vec{a} \in D_w, w : p(\vec{a}), \Gamma \Rightarrow \Delta, w : p(\vec{a})}\,(id_q^*)$$

$$\frac{\mathcal{R}, a \in D_w, \Gamma \Rightarrow \Delta, w : A[a/x], w : \exists x A}{\mathcal{R}, a \in D_w, \Gamma \Rightarrow \Delta, w : \exists x A}\,(\exists_r)$$

$$\frac{\mathcal{R}, w \leq v, a \in D_v, v : A[a/x], w : \forall x A, \Gamma \Rightarrow \Delta}{\mathcal{R}, w \leq v, a \in D_v, w : \forall x A, \Gamma \Rightarrow \Delta}\,(\forall_l)$$

# Eliminating Domain Conditions: $(nd)$ Case

$$\dfrac{\dfrac{\mathcal{R}, u \leq w, a \in D_u, a \in D_w, \Gamma \Rightarrow \Delta, w : A[a/x], w : \exists x A}{\mathcal{R}, u \leq w, a \in D_u, a \in D_w, \Gamma \Rightarrow \Delta, w : \exists x A}\,(\exists_r)}{\mathcal{R}, u \leq w, a \in D_u, \Gamma \Rightarrow \Delta, w : \exists x A}\,(nd)$$

$$\dfrac{\mathcal{R}, u \leq w, a \in D_u, a \in D_w, \Gamma \Rightarrow \Delta, w : A[a/x], w : \exists x A}{\mathcal{R}, u \leq w, a \in D_u, \Gamma \Rightarrow \Delta, w : A[a/x], w : \exists x A}\,(nd)$$

# Eliminating Domain Conditions: $(nd)$ Case

$$\cfrac{\cfrac{\mathcal{R}, u \leq w, a \in D_u, a \in D_w, \Gamma \Rightarrow \Delta, w : A[a/x], w : \exists x A}{\mathcal{R}, u \leq w, a \in D_u, a \in D_w, \Gamma \Rightarrow \Delta, w : \exists x A}\,(\exists_r)}{\mathcal{R}, u \leq w, a \in D_u, \Gamma \Rightarrow \Delta, w : \exists x A}\,(nd)$$

$$\cfrac{\mathcal{R}, u \leq w, a \in D_u, a \in D_w, \Gamma \Rightarrow \Delta, w : A[a/x], w : \exists x A}{\mathcal{R}, u \leq w, a \in D_u, \Gamma \Rightarrow \Delta, w : A[a/x], w : \exists x A}\,(nd)$$

*Important Observation: The domain atom has *shifted one step back*.

# Eliminating Domain Conditions: $(cd)$ Case

$$\dfrac{\dfrac{\mathcal{R}, w \leq v, a \in D_w, a \in D_v, \Gamma \Rightarrow \Delta, w : A[a/x], w : \exists x A}{\mathcal{R}, w \leq v, a \in D_w, a \in D_v, \Gamma \Rightarrow \Delta, w : \exists x A} (\exists_r)}{\mathcal{R}, w \leq v, a \in D_v, \Gamma \Rightarrow \Delta, w : \exists x A} (cd)$$

$$\dfrac{\mathcal{R}, w \leq v, a \in D_w, a \in D_v, \Gamma \Rightarrow \Delta, w : A[a/x], w : \exists x A}{\mathcal{R}, w \leq v, a \in D_v, \Gamma \Rightarrow \Delta, w : A[a/x], w : \exists x A} (cd)$$

# Eliminating Domain Conditions: $(cd)$ Case

$$\dfrac{\dfrac{\mathcal{R}, w \leq v, a \in D_w, a \in D_v, \Gamma \Rightarrow \Delta, w : A[a/x], w : \exists x A}{\mathcal{R}, w \leq v, a \in D_w, a \in D_v, \Gamma \Rightarrow \Delta, w : \exists x A}\ (\exists_r)}{\mathcal{R}, w \leq v, a \in D_v, \Gamma \Rightarrow \Delta, w : \exists x A}\ (cd)$$

$$\dfrac{\mathcal{R}, w \leq v, a \in D_w, a \in D_v, \Gamma \Rightarrow \Delta, w : A[a/x], w : \exists x A}{\mathcal{R}, w \leq v, a \in D_v, \Gamma \Rightarrow \Delta, w : A[a/x], w : \exists x A}\ (cd)$$

*Important Observation: The domain atom has *shifted one step forward*.

# Newly Added Rules

- $\dagger_1$ = Path between $v$ and $w$
- $\dagger_2$ = Path between $v_i$ and $w$ for $1 \leq i \leq n$

$$\frac{\mathcal{R}, a \in D_v, \Gamma \Rightarrow \Delta, w : A[a/x], w : \exists x A}{\mathcal{R}, a \in D_v, \Gamma \Rightarrow \Delta, w : \exists x A} \; (\exists^*_r)^{\dagger_1}$$

$$\frac{\mathcal{R}, a \in D_v, w : A[a/x], w : \forall x A, \Gamma \Rightarrow \Delta}{\mathcal{R}, a \in D_v, w : \forall x A, \Gamma \Rightarrow \Delta} \; (\forall^*_l)^{\dagger_1}$$

$$\frac{}{\mathcal{R}, a_1 \in D_{v_1}, \ldots, a_n \in D_{v_n}, w : p(\vec{a}), \Gamma \Rightarrow w : p(\vec{a}), \Delta} \; (id^*_q)^{\dagger_2}$$

# Inherited Properties

NIntQC has the following properties:

- Admissibility of the following structural rules:

$$\frac{\Sigma\{X \rightarrow Y, [\Sigma'], [\Sigma']\}}{\Sigma\{X \rightarrow Y, [\Sigma']\}}\,(ctr_1) \quad \frac{\Sigma\{X, X \rightarrow Y\}}{\Sigma\{X \rightarrow Y\}}\,(ctr_2)$$

$$\frac{\Sigma\{X \rightarrow Y, Y\}}{\Sigma\{X \rightarrow Y\}}\,(ctr_3) \quad \frac{\Sigma\{X \rightarrow Y\}}{\Sigma\{X \rightarrow Y, [\Sigma']\}}\,(wk_1)$$

$$\frac{\Sigma\{X \rightarrow Y\}}{\Sigma\{X \rightarrow Y, Z\}}\,(wk_2) \quad \frac{\Sigma\{X \rightarrow Y\}}{\Sigma\{X, Z \rightarrow Y\}}\,(wk_3)$$

- Invertibility of all rules

- Sound and Cut-free Complete

# Future Work

1. Find General Conditions for Extracting Nested from Labelled
2. Current Results:
    - (Multi-Modal) Grammar Logics
    - (Sub-)(Bi-)Intuitionistic Logics
    - STIT Logics
3. Provide Nested Calculi for New Logics