Relating Labelled and Label-Free Bunched Calculi in BI logic

D. Galmiche - M. Marti - D. Méry

Université de Lorraine, CNRS, LORIA
Nancy - France

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Overview

Introduction

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BI (Bunched Implication) Logic

Sharing and separation of resources

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BI (Bunched Implication) Logic

Sharing and separation of resources

Formulas given by

$$A \ ::= \ p \mid \top_{\mathfrak{m}} \mid A \mathrel{-\!\!*} A \mid A * A \mid \bot \mid \top_{\mathfrak{a}} \mid A \to A \mid A \wedge A \mid A \vee A$$

BI (Bunched Implication) Logic

Sharing and separation of resources

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$$\mathrm{A} \ ::= \ \mathrm{p} \mid \top_{\mathfrak{m}} \mid \mathrm{A} \mathrel{-\!\!*} \mathrm{A} \mid \mathrm{A} * \mathrm{A} \mid \bot \mid \top_{\mathfrak{a}} \mid \mathrm{A} \to \mathrm{A} \mid \mathrm{A} \wedge \mathrm{A} \mid \mathrm{A} \vee \mathrm{A}$$

Elementary semantics

Preordered monoid of resources $(M, \bullet, \sqsubseteq)$ + forcing relation

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Preordered monoid of resources $(M, \bullet, \sqsubseteq)$ + forcing relation

$$m \models \mathrm{A} \wedge \mathrm{B} \quad \Longleftrightarrow \quad m \models \mathrm{A} \text{ and } m \models \mathrm{B}$$

$$m \models \mathrm{A} * \mathrm{B} \quad \Longleftrightarrow \quad \exists n, n' \in M.\ n \bullet n' \sqsubseteq m, n \models \mathrm{A} \text{ and } n' \models \mathrm{B}$$

BI (Bunched Implication) Logic

Sharing and separation of resources

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$$\mathrm{A} \quad ::= \quad \mathrm{p} \mid \top_{\mathfrak{m}} \mid \mathrm{A} \mathrel{-\!\!*} \mathrm{A} \mid \mathrm{A} * \mathrm{A} \mid \bot \mid \top_{\mathfrak{a}} \mid \mathrm{A} \rightarrow \mathrm{A} \mid \mathrm{A} \wedge \mathrm{A} \mid \mathrm{A} \vee \mathrm{A}$$

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Preordered monoid of resources $(M, \bullet, \sqsubseteq)$ + forcing relation

$$m \models \mathrm{A} \wedge \mathrm{B} \iff m \models \mathrm{A} \text{ and } m \models \mathrm{B}$$

$$m \models \mathrm{A} * \mathrm{B} \iff \exists n, n' \in M.\ n \bullet n' \sqsubseteq m, n \models \mathrm{A} \text{ and } n' \models \mathrm{B}$$

$$m \models \mathrm{A} \rightarrow \mathrm{B} \iff \forall n \in M.\ \text{if } m \sqsubseteq n \text{ and } n \models \mathrm{A} \text{ then } n \models \mathrm{B}$$

$$m \models \mathrm{A} \mathrel{-\!\!*} \mathrm{B} \iff \forall n \in M.\ \text{if } n \models \mathrm{A} \text{ then } m \bullet n \models \mathrm{B}$$

Introduction

Label-Free Proof Systems

Label-Free Proof Systems

- Sequent Calculus LBI (D. Pym, 2002)
- Natural Deduction NBI (D. Pym, 2002)
- Display Calculus (for BBI, J. Brotherston, 2012)

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Labelled Proof Systems

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Label-Free Proof Systems

- Sequent Calculus LBI (D. Pym, 2002)
- Natural Deduction NBI (D. Pym, 2002)
- Display Calculus (for BBI, J. Brotherston, 2012)

Labelled Proof Systems

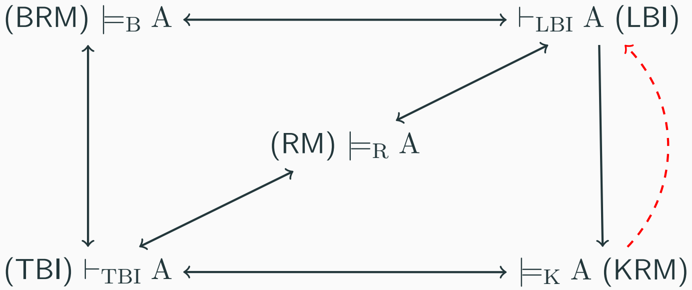
- Tableaux TBI (for BI, D. Galmiche and D. Mery, 2005)
- Tableaux (for BBI, D. Larchey, 2014)
- Sequent Calculus (for BBI, Z. Hòu, A. Tiu and R. Goré, 2013)

Disjunction and Completeness

- Beth: $m \models \mathrm{A} \vee \mathrm{B}$ iff $\exists n, n'.\ n \sqcap n' \sqsubseteq m, n \models \mathrm{A}$ and $n' \models \mathrm{B}$
- Kripke: $m \models \mathrm{A} \vee \mathrm{B}$ iff $m \models \mathrm{A}$ or $m \models \mathrm{B}$

completeness wrt LBI known for relational models

still unknown for monoidal models



Introduction

Sequents

$$\Gamma \vdash A \quad \text{with} \quad \Gamma \text{ a bunch, } A \text{ a formula}$$

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$$\Gamma \;::=\; \varnothing_{\mathfrak{a}} \mid \varnothing_{\mathfrak{m}} \mid \mathrm{A} \mid \Gamma\,;\Gamma \mid \Gamma\,,\Gamma$$

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$$\varnothing_{\mathfrak{m}}\,,(\mathrm{A} \to \mathrm{B}\,;\mathrm{C}\,;(\mathrm{A}\,,\mathrm{B} \wedge \mathrm{C}\,,\varnothing_{\mathfrak{a}}))$$

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$$\varnothing_{\mathfrak{m}}\,,(A \to B\,;C\,;(A\,,B \land C\,,\varnothing_{\mathfrak{a}}))$$

$$\frac{\Gamma\;;\;A \vdash B}{\Gamma \vdash A \to B} \qquad \frac{\Gamma\;,\;A \vdash B}{\Gamma \vdash A \multimap B}$$

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$$\Gamma \quad ::= \quad \varnothing_{\mathfrak{a}} \mid \varnothing_{\mathfrak{m}} \mid A \mid \Gamma \, ; \Gamma \mid \Gamma \, , \Gamma$$

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$$\frac{\Gamma \land A \vdash B}{\Gamma \vdash A \to B} \qquad \frac{\Gamma * A \vdash B}{\Gamma \vdash A \mathrel{-\!\!*} B}$$

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Question

How does LBI relates to labelled calculi ?

Introduction

Contributions

- An alternative resource semantics
- GBI, a new labelled sequent calculus

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- Translation of LBI-proofs into GBI-proofs
    - not a one-to-one correspondence
    - additional structural steps

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Contributions

- An alternative resource semantics
- GBI, a new labelled sequent calculus
- Translation of LBI-proofs into GBI-proofs
    - not a one-to-one correspondence
    - additional structural steps
- Translation of GBI-proofs into LBI-proofs
    - restriction of GBI to one single conclusion
    - GBI-proofs satisfying a tree property

The Logic BI

BI logic

BI logic

Formulas

$$A \quad ::= \quad p \mid \top_{\mathfrak{m}} \mid A \mathrel{-\!\!*} A \mid A * A \mid \bot \mid \top_{\mathfrak{a}} \mid A \to A \mid A \wedge A \mid A \vee A$$

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$\Gamma(\Delta) \iff \Delta$ is a sub-tree of Γ

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$$\Gamma \quad ::= \quad \varnothing_{\mathfrak{a}} \mid \varnothing_{\mathfrak{m}} \mid A \mid \Gamma ; \Gamma \mid \Gamma , \Gamma$$

$\Gamma(\Delta) \iff \Delta$ is a sub-tree of Γ

$\Gamma \equiv \Delta \iff \Gamma$ equal to Δ upto

- associativity and commutativity of ; and ,
- identity of $\varnothing_{\mathfrak{a}}$ wrt " ; " and $\varnothing_{\mathfrak{m}}$ wrt " , "

Structural Equivalence

$$\frac{\Gamma \vdash A}{\Delta \vdash A}\ \Gamma{\equiv}\Delta$$

LBI Sequent Calculus

Structural Equivalence

- Commutativity, Associativity, Identity wrt. Units

$$\frac{\Gamma(\Delta_1\,;\Delta_2) \vdash A}{\Gamma(\Delta_2\,;\Delta_1) \vdash A}\ \mathrm{E_a} \qquad \frac{\Gamma(\Delta_1\,,\Delta_2) \vdash A}{\Gamma(\Delta_2\,,\Delta_1) \vdash A}\ \mathrm{E_m}$$

$$\frac{\Gamma((\Delta_1\,;\Delta_2)\,;\Delta_3) \vdash A}{\Gamma(\Delta_2\,;(\Delta_1\,;\Delta_3)) \vdash A}\ \mathrm{A_a} \qquad \frac{\Gamma((\Delta_1\,,\Delta_2)\,,\Delta_3) \vdash A}{\Gamma(\Delta_2\,,(\Delta_1\,,\Delta_3)) \vdash A}\ \mathrm{A_m}$$

$$\frac{\Gamma(\Delta) \vdash A}{\Gamma(\varnothing_{\mathfrak{a}}\,;\Delta) \vdash A}\ \mathrm{U_a} \qquad \frac{\Gamma(\Delta) \vdash A}{\Gamma(\varnothing_{\mathfrak{m}}\,,\Delta) \vdash A}\ \mathrm{U_m}$$

LBI Sequent Calculus

Structural Rules

$$\frac{\Gamma(\Delta_1) \vdash A}{\Gamma(\Delta_1 \,;\, \Delta_2) \vdash A}\ \mathsf{wk} \qquad \frac{\Gamma(\Delta \,;\, \Delta) \vdash A}{\Gamma(\Delta) \vdash A}\ \mathsf{ctr}$$

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- Apply only to " ; " not to " , "

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- Apply only to " ; " not to " , "

$$\frac{\Gamma(C) \vdash A \qquad \Delta \vdash C}{\Gamma(\Delta) \vdash A}\ \mathsf{cut}$$

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- Apply only to " ; " not to " , "

$$\frac{\Gamma(C) \vdash A \qquad \Delta \vdash C}{\Gamma(\Delta) \vdash A}\ \mathsf{cut}$$

- Cut-elimination holds in LBI

LBI Sequent Calculus

Axioms

$$\frac{}{A \vdash A}\ \mathsf{id} \qquad \frac{}{\Gamma(\bot) \vdash A}\ \bot_L$$

$$\frac{}{\varnothing_{\mathfrak{a}} \vdash \top_{\mathfrak{a}}}\ \top_{\mathfrak{a}} R \qquad \frac{}{\varnothing_{\mathfrak{m}} \vdash \top_{\mathfrak{m}}}\ \top_{\mathfrak{m}} R$$

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Units

$$\frac{\Gamma(\varnothing_{\mathfrak{a}}) \vdash A}{\Gamma(\top_{\mathfrak{a}}) \vdash A}\ \top_{\mathfrak{a}} L \qquad \frac{\Gamma(\varnothing_{\mathfrak{m}}) \vdash A}{\Gamma(\top_{\mathfrak{m}}) \vdash A}\ \top_{\mathfrak{m}} L$$

Additive Logical Rules

$$\frac{\Gamma(A \,;B) \vdash C}{\Gamma(A \wedge B) \vdash C} \wedge_L \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma \,; \Delta \vdash A \wedge B} \wedge_R$$

$$\frac{\Delta \vdash A \quad \Gamma(B) \vdash C}{\Gamma(\Delta \,; A \to B) \vdash C} \to_L \qquad \frac{\Gamma \,; A \vdash B}{\Gamma \vdash A \to B} \to_R$$

$$\frac{\Gamma(A) \vdash C \quad \Gamma(B) \vdash C}{\Gamma(A \vee B) \vdash C} \vee_L \qquad \frac{\Gamma \vdash A_{i \in \{1,2\}}}{\Gamma \vdash A_1 \vee A_2} \vee_{R_i}$$

Multiplicative Logical Rules

$$\frac{\Gamma(A\,,B) \vdash C}{\Gamma(A * B) \vdash C} *_L \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma\,,\Delta \vdash A * B} *_R$$

$$\frac{\Delta \vdash A \quad \Gamma(B) \vdash C}{\Gamma(\Delta\,,A \mathrel{-\!\!*} B) \vdash C} \mathrel{-\!\!*}_L \qquad \frac{\Gamma\,,A \vdash B}{\Gamma \vdash A \mathrel{-\!\!*} B} \mathrel{-\!\!*}_R$$

LBI Sequent Calculus

Definition

A formula C is a theorem of LBI iff $\varnothing_{\mathfrak{m}} \vdash \mathrm{C}$ is provable in LBI.

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LBI-Proof of $(\mathrm{A} * (\mathrm{A} \mathrel{-\!\!*} \mathrm{B})) \mathrel{-\!\!*} \mathrm{B}$

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$$\frac{}{\varnothing_{\mathfrak{m}} \vdash (\mathrm{A} * (\mathrm{A} \mathrel{-\!\!*} \mathrm{B})) \mathrel{-\!\!*} \mathrm{B}} \; {-\!\!*}_{\mathrm{L}}$$

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$$\dfrac{\dfrac{}{\varnothing_{\mathfrak{m}}, \mathrm{A} * (\mathrm{A} \mathrel{-\!\!*} \mathrm{B}) \vdash \mathrm{B}} \equiv}{\varnothing_{\mathfrak{m}} \vdash (\mathrm{A} * (\mathrm{A} \mathrel{-\!\!*} \mathrm{B})) \mathrel{-\!\!*} \mathrm{B}} \mathrel{-\!\!*}_{\mathrm{L}}$$

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A formula C is a theorem of LBI iff $\varnothing_{\mathfrak{m}} \vdash \mathrm{C}$ is provable in LBI.

LBI-Proof of $(\mathrm{A} * (\mathrm{A} \mathrel{-\!\!*} \mathrm{B})) \mathrel{-\!\!*} \mathrm{B}$

$$\dfrac{\dfrac{\overline{\mathrm{A} * (\mathrm{A} \mathrel{-\!\!*} \mathrm{B}) \vdash \mathrm{B}}\ {*}_{\mathrm{L}}}{\varnothing_{\mathfrak{m}} , \mathrm{A} * (\mathrm{A} \mathrel{-\!\!*} \mathrm{B}) \vdash \mathrm{B}}\ \equiv}{\varnothing_{\mathfrak{m}} \vdash (\mathrm{A} * (\mathrm{A} \mathrel{-\!\!*} \mathrm{B})) \mathrel{-\!\!*} \mathrm{B}}\ {\mathrel{-\!\!*}}_{\mathrm{L}}$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{A}\,,\mathrm{A} \mathrel{-\!\!*} \mathrm{B} \vdash \mathrm{B}}\ {-\!\!*}_{\mathrm{L}}}{\mathrm{A} * (\mathrm{A} \mathrel{-\!\!*} \mathrm{B}) \vdash \mathrm{B}}\ *_{\mathrm{L}}}{\varnothing_{\mathfrak{m}}\,, \mathrm{A} * (\mathrm{A} \mathrel{-\!\!*} \mathrm{B}) \vdash \mathrm{B}}\ \equiv}{\varnothing_{\mathfrak{m}} \vdash (\mathrm{A} * (\mathrm{A} \mathrel{-\!\!*} \mathrm{B})) \mathrel{-\!\!*} \mathrm{B}}\ {-\!\!*}_{\mathrm{L}}$$

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A formula C is a theorem of LBI iff $\varnothing_{\mathfrak{m}} \vdash \mathrm{C}$ is provable in LBI.

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$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{A} \vdash \mathrm{A}}\ \mathrm{id}}{\mathrm{A}\, ,\, \mathrm{A} \mathrel{-\!\!*} \mathrm{B} \vdash \mathrm{B}}\ {-\!\!*}_{\mathrm{L}}}{\mathrm{A} * (\mathrm{A} \mathrel{-\!\!*} \mathrm{B}) \vdash \mathrm{B}}\ *_{\mathrm{L}}}{\varnothing_{\mathfrak{m}}\, ,\, \mathrm{A} * (\mathrm{A} \mathrel{-\!\!*} \mathrm{B}) \vdash \mathrm{B}}\ \equiv}{\varnothing_{\mathfrak{m}} \vdash (\mathrm{A} * (\mathrm{A} \mathrel{-\!\!*} \mathrm{B})) \mathrel{-\!\!*} \mathrm{B}}\ {-\!\!*}_{\mathrm{L}}
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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{A} \vdash \mathrm{A}}\ \mathrm{id} \qquad \dfrac{}{\mathrm{B} \vdash \mathrm{B}}\ \mathrm{id}}{\mathrm{A} \,,\, \mathrm{A} \mathrel{-\!*} \mathrm{B} \vdash \mathrm{B}}\ {-\!*}_{\mathrm{L}}}{\mathrm{A} * (\mathrm{A} \mathrel{-\!*} \mathrm{B}) \vdash \mathrm{B}}\ *_{\mathrm{L}}}{\varnothing_{\mathfrak{m}} \,,\, \mathrm{A} * (\mathrm{A} \mathrel{-\!*} \mathrm{B}) \vdash \mathrm{B}}\ \equiv}{\varnothing_{\mathfrak{m}} \vdash (\mathrm{A} * (\mathrm{A} \mathrel{-\!*} \mathrm{B})) \mathrel{-\!*} \mathrm{B}}\ {-\!*}_{\mathrm{L}}$$

LBI Sequent Calculus

Semi Distributivity

$$\frac{\Gamma((\Delta_1\,,\Delta_2)\,;(\Delta_1\,,\Delta_3)) \vdash \mathrm{C}}{\Gamma(\Delta_1\,,(\Delta_2\,;\Delta_3)) \vdash \mathrm{C}}\ \mathrm{sd}$$

LBI Sequent Calculus

Semi Distributivity

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Lemma

Semi-distributivity is derivable in LBI.

Semi Distributivity

$$\frac{\Gamma((\Delta_1\,,\Delta_2)\,;(\Delta_1\,,\Delta_3))\vdash \mathrm{C}}{\Gamma(\Delta_1\,,(\Delta_2\,;\Delta_3))\vdash \mathrm{C}}\ \mathrm{sd}$$

Lemma

Semi-distributivity is derivable in LBI.

Definition

$\mathrm{LBI}_{\mathsf{sd}} = \mathrm{LBI} + \mathrm{sd} +$ contraction restricted to $\top_{\mathbb{m}}$.

LBI Sequent Calculus

Semi Distributivity

$$\frac{\Gamma((\Delta_1\,,\Delta_2)\,;(\Delta_1\,,\Delta_3))\vdash \mathrm{C}}{\Gamma(\Delta_1\,,(\Delta_2\,;\Delta_3))\vdash \mathrm{C}}\ \mathrm{sd}$$

Lemma

Semi-distributivity is derivable in LBI.

Definition

$$\mathrm{LBI}_{\mathsf{sd}} = \mathrm{LBI} + \mathrm{sd} + \text{contraction restricted to } \top_{\mathfrak{m}}.$$

Lemma

LBI without contraction of bunches is not complete !

BI: an Alternative Semantics

Resource Monoid $\mathcal{M} = (M, \otimes, 1, \oplus, 0, \infty, \sqsubseteq)$

- $(M, \otimes, 1)$ and $(M, \oplus, 0)$ commutative monoids
- $\sqsubseteq$ partial order on M where $\forall m, n \in M$
  - $0 \sqsubseteq m$ and $m \sqsubseteq \infty$
  - $m \sqsubseteq m \oplus n$ and $m \oplus m \sqsubseteq m$
  - $\infty \sqsubseteq \infty \otimes m$ (and $\infty \sqsubseteq \infty \oplus m$)
- $\otimes$ and $\oplus$ bifunctorial (compatible) wrt. $\sqsubseteq$

  $m \sqsubseteq n$ and $m' \sqsubseteq n' \implies m \otimes m' \sqsubseteq n \otimes n'$ and $m \oplus m' \sqsubseteq n \oplus n'$

BI: an Alternative Semantics

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Resource Interpretation $[\![-]\!] : Prop \longrightarrow \mathcal{P}(M)$

- $\forall m, n \in M.\ m \sqsubseteq n$ and $m \in [\![\mathrm{p}]\!] \Longrightarrow n \in [\![\mathrm{p}]\!]$
- $\forall \mathrm{p}.\ \infty \in [\![\mathrm{p}]\!]$

BI: an Alternative Semantics

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Resource model (RM) $\mathcal{K} = (\mathcal{M}, [\![-]\!], \models)$

- $\mathcal{M}$ resource monoid, $[\![-]\!]$ resource interpretation

BI: an Alternative Semantics

Resource model (RM) $\mathcal{K} = (\mathcal{M}, [\![-]\!], \models)$

- $\mathcal{M}$ resource monoid, $[\![-]\!]$ resource interpretation
- $\models$ is a forcing relation such that

$$
\begin{aligned}
m \models \mathrm{p} &\Leftrightarrow m \in [\![\mathrm{p}]\!] \\
m \models \top_{\mathfrak{a}} &\Leftrightarrow 0 \sqsubseteq m \text{ (always)} \\
m \models \top_{\mathfrak{m}} &\Leftrightarrow 1 \sqsubseteq m \\
m \models \bot &\Leftrightarrow \infty \sqsubseteq m \\
m \models \mathrm{A} \land \mathrm{B} &\Leftrightarrow \exists n, n' \in M.\ n \oplus n' \sqsubseteq m, n \models \mathrm{A} \text{ and } n' \models \mathrm{B} \\
m \models \mathrm{A} * \mathrm{B} &\Leftrightarrow \exists n, n' \in M.\ n \otimes n' \sqsubseteq m, n \models \mathrm{A} \text{ and } n' \models \mathrm{B} \\
m \models \mathrm{A} \to \mathrm{B} &\Leftrightarrow \forall n, n' \in M.\ n \models \mathrm{A} \text{ and } m \oplus n \sqsubseteq n' \Rightarrow n' \models \mathrm{B} \\
m \models \mathrm{A} \mathrel{-\!\!*} \mathrm{B} &\Leftrightarrow \forall n, n' \in M.\ n \models \mathrm{A} \text{ and } m \otimes n \sqsubseteq n' \Rightarrow n' \models \mathrm{B}
\end{aligned}
$$

GBI - a Labelled Calculus for BI

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Labels and Constraints

- atomic labels: $\{\alpha, \beta, \gamma, \ldots\}\{0, 1\}^* \cup \{\mathrm{m}, \mathrm{a}, \varpi\}$
- labels: $\mathfrak{a}(\ell_1, \ell_2)$ or $\mathfrak{m}(\ell_1, \ell_2)$, where ℓ_1, ℓ_2 are labels
  $\mathfrak{a}$ mimics $\oplus$, $\mathfrak{m}$ mimics $\otimes$
- label constraints: $\ell \leqslant \ell'$

GBI - a Labelled Calculus for BI

Labels and Constraints

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Labelled formulas $\mathrm{A} : \ell$

- with A a BI formula and ℓ a label

GBI - a Labelled Calculus for BI

Labels and Constraints

- atomic labels: $\{\,\alpha, \beta, \gamma, \ldots\,\}\{\,0, 1\,\}^* \cup \{\,\mathrm{m}, \mathrm{a}, \varpi\,\}$
- labels: $\mathfrak{a}(\ell_1, \ell_2)$ or $\mathfrak{m}(\ell_1, \ell_2)$, where ℓ_1, ℓ_2 are labels
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- label constraints: $\ell \leqslant \ell'$

Labelled formulas $\mathrm{A} : \ell$

- with A a BI formula and ℓ a label

$\mathbf{GBI}$ sequents $\Gamma \vdash \Delta$

- Γ set of labels, label constraints and labelled formulas
- Δ set of labelled formulas

GBI – Axioms and Units

$$\frac{}{\Gamma, A : \ell \vdash A : \ell, \Delta}\ \mathrm{id}$$

$$\frac{\Gamma, \varpi \leqslant \ell \vdash \Delta}{\Gamma, \bot : \ell \vdash \Delta}\ \bot_{\mathrm{L}} \qquad \frac{}{\Gamma, \varpi \leqslant \ell \vdash A : \ell, \Delta}\ \bot_{\mathrm{R}}$$

$$\frac{\Gamma, \mathrm{m} \leqslant \ell \vdash \Delta}{\Gamma, \top_{\mathfrak{m}} : \ell \vdash \Delta}\ \top_{\mathfrak{m}}\mathrm{L} \qquad \frac{}{\Gamma, \mathrm{m} \leqslant \ell \vdash \top_{\mathfrak{m}} : \ell, \Delta}\ \top_{\mathfrak{m}}\mathrm{R}$$

$$\frac{\Gamma, \mathrm{a} \leqslant \ell \vdash \Delta}{\Gamma, \top_{\mathfrak{a}} : \ell \vdash \Delta}\ \top_{\mathfrak{a}}\mathrm{L} \qquad \frac{}{\Gamma, \mathrm{a} \leqslant \ell \vdash \top_{\mathfrak{a}} : \ell, \Delta}\ \top_{\mathfrak{a}}\mathrm{R}$$

GBI – Implication rules

- In $-\!\!*_{\mathrm{R}}$ and $\rightarrow_{\mathrm{R}}$: ℓ_1, ℓ_2 must be fresh atomic labels

$$\frac{\mathfrak{a}(\ell, \ell_1) \leqslant \ell_2, \Gamma, \mathrm{A} : \ell_1 \vdash \mathrm{B} : \ell_2, \Delta}{\Gamma \vdash \mathrm{A} \rightarrow \mathrm{B} : \ell, \Delta} \rightarrow_{\mathrm{R}}$$

$$\frac{\mathfrak{m}(\ell, \ell_1) \leqslant \ell_2, \Gamma, \mathrm{A} : \ell_1 \vdash \mathrm{B} : \ell_2, \Delta}{\Gamma \vdash \mathrm{A} -\!\!* \mathrm{B} : \ell, \Delta} -\!\!*_{\mathrm{R}}$$

$$\frac{\mathfrak{a}(\ell, \ell_1) \leqslant \ell_2, \Gamma \vdash \mathrm{A} : \ell_1, \Delta \qquad \mathfrak{a}(\ell, \ell_1) \leqslant \ell_2, \Gamma, \mathrm{B} : \ell_2 \vdash \Delta}{\mathfrak{a}(\ell, \ell_1) \leqslant \ell_2, \Gamma, \mathrm{A} \rightarrow \mathrm{B} : \ell \vdash \Delta} \rightarrow_{\mathrm{L}}$$

$$\frac{\mathfrak{m}(\ell, \ell_1) \leqslant \ell_2, \Gamma \vdash \mathrm{A} : \ell_1, \Delta \qquad \mathfrak{m}(\ell, \ell_1) \leqslant \ell_2, \Gamma, \mathrm{B} : \ell_2 \vdash \Delta}{\mathfrak{m}(\ell, \ell_1) \leqslant \ell_2, \Gamma, \mathrm{A} -\!\!* \mathrm{B} : \ell \vdash \Delta} -\!\!*_{\mathrm{L}}$$

GBI – Conjunction Rules

- In $*_{\mathrm{L}}$ and $\wedge_{\mathrm{L}}$: ℓ_1, ℓ_2 must be fresh atomic labels

$$\frac{\mathfrak{a}(\ell_1, \ell_2) \leqslant \ell, \Gamma, \mathrm{A} : \ell_1, \mathrm{B} : \ell_2 \vdash \Delta}{\Gamma, \mathrm{A} \wedge \mathrm{B} : \ell \vdash \Delta} \wedge_{\mathrm{L}}$$

$$\frac{\mathfrak{m}(\ell_1, \ell_2) \leqslant \ell, \Gamma, \mathrm{A} : \ell_1, \mathrm{B} : \ell_2 \vdash \Delta}{\Gamma, \mathrm{A} * \mathrm{B} : \ell \vdash \Delta} *_{\mathrm{L}}$$

$$\frac{\mathfrak{a}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \mathrm{A} : \ell_1, \Delta \qquad \mathfrak{a}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \mathrm{B} : \ell_2, \Delta}{\mathfrak{a}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \mathrm{A} \wedge \mathrm{B} : \ell, \Delta} \wedge_{\mathrm{R}}$$

$$\frac{\mathfrak{m}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \mathrm{A} : \ell_1, \Delta \qquad \mathfrak{m}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \mathrm{B} : \ell_2, \Delta}{\mathfrak{m}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \mathrm{A} * \mathrm{B} : \ell, \Delta} *_{\mathrm{R}}$$

GBI – Monotonicity, Weakening and Contraction

$$\frac{\ell_1 \leqslant \ell_2, \Gamma, \mathrm{A} : \ell_2 \vdash \Delta}{\ell_1 \leqslant \ell_2, \Gamma, \mathrm{A} : \ell_1 \vdash \Delta}\,\mathrm{K_L} \qquad \frac{\ell_1 \leqslant \ell_0, \Gamma \vdash \mathrm{C} : \ell_1, \Delta}{\ell_1 \leqslant \ell_0, \Gamma \vdash \mathrm{C} : \ell_0, \Delta}\,\mathrm{K_R}$$

$$\frac{\Gamma_0 \vdash \Delta}{\Gamma_0, \Gamma_1 \vdash \Delta}\,\mathrm{W_L} \qquad \frac{\Gamma_0, \Gamma_1, \Gamma_1 \vdash \Delta}{\Gamma_0, \Gamma_1 \vdash \Delta}\,\mathrm{C_L}$$

$$\frac{\Gamma \vdash \Delta_0}{\Gamma \vdash \Delta_0, \Delta_1}\,\mathrm{W_R} \qquad \frac{\Gamma \vdash \Delta_0, \Delta_1, \Delta_1}{\Gamma \vdash \Delta_0, \Delta_1, \Delta_1}\,\mathrm{C_R}$$

GBI – Structural Rules

- ℓ in R and $\mathrm{I}_{\mathfrak{a}}$ must occur in Γ, Δ or $\{\, \mathrm{m}, \mathrm{a}, \varpi \,\}$

$$\frac{\ell \leqslant \ell, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ \mathrm{R} \qquad \frac{\ell_0 \leqslant \ell, \ell_0 \leqslant \ell_1, \ell_1 \leqslant \ell, \Gamma \vdash \Delta}{\ell_0 \leqslant \ell_1, \ell_1 \leqslant \ell, \Gamma \vdash \Delta}\ \mathrm{T}$$

$$\frac{\mathfrak{r}(\ell_2, \ell_1) \leqslant \ell, \Gamma \vdash \Delta}{\mathfrak{r}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \Delta}\ \mathrm{E}_{\mathfrak{r}} \qquad \frac{\mathfrak{a}(\ell, \ell) \leqslant \ell, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ \mathrm{I}_{\mathfrak{a}}$$

- ℓ in $\mathrm{A}_{\mathfrak{r}}^{\mathrm{i} \in \{\, 1,2 \,\}}$ is a fresh atomic label

$$\frac{\mathfrak{r}(\ell_3, \ell_2) \leqslant \ell_0, \mathfrak{r}(\ell_4, \ell_0) \leqslant \ell, \Gamma \vdash \Delta}{\mathfrak{r}(\ell_4, \ell_3) \leqslant \ell_1, \mathfrak{r}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \Delta}\ \mathrm{A}_{\mathfrak{r}}^1$$

$$\frac{\mathfrak{r}(\ell_1, \ell_4) \leqslant \ell_0, \mathfrak{r}(\ell_0, \ell_3) \leqslant \ell, \Gamma \vdash \Delta}{\mathfrak{r}(\ell_4, \ell_3) \leqslant \ell_2, \mathfrak{r}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \Delta}\ \mathrm{A}_{\mathfrak{r}}^2$$

GBI – Structural Rules

$$\frac{\mathfrak{r}(\ell, \mathrm{r}) \leqslant \ell, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ \mathrm{U}^1_{\mathfrak{r}} \qquad \frac{\mathfrak{r}(\mathrm{r}, \ell) \leqslant \ell, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ \mathrm{U}^2_{\mathfrak{r}}$$

$$\frac{\mathfrak{r}(\ell_0, \ell_2) \leqslant \ell, \ell_0 \leqslant \ell_1, \mathfrak{r}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \Delta}{\ell_0 \leqslant \ell_1, \mathfrak{r}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \Delta}\ \mathrm{C}^1_{\mathfrak{r}}$$

$$\frac{\mathfrak{r}(\ell_1, \ell_0) \leqslant \ell, \ell_0 \leqslant \ell_2, \mathfrak{r}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \Delta}{\ell_0 \leqslant \ell_2, \mathfrak{r}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \Delta}\ \mathrm{C}^2_{\mathfrak{r}}$$

- ℓ_{3-i} in $\mathrm{P}^{\mathrm{i}}_{\mathfrak{m}}$ must be in $\{\, \mathrm{m}, \varpi \,\}$

$$\frac{\ell_i \leqslant \ell, \mathfrak{r}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \Delta}{\mathfrak{r}(\ell_1, \ell_2) \leqslant \ell, \Gamma \vdash \Delta}\ \mathrm{P}^{\mathrm{i}}_{\mathfrak{r}}$$

GBI – Example

Definition

A formula C is a theorem of GBI iff $\vdash \mathrm{C} : \mathrm{m}$ provable in GBI

An example

$$
\dfrac{
\dfrac{\dfrac{}{\mathrm{A} : \ell_3 \vdash \mathrm{A} : \ell_3}\ \mathrm{id}}{-, \mathrm{A} : \ell_3 \vdash \mathrm{A} : \ell_3}\ \mathrm{W_L}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{B} : \ell_1 \vdash \mathrm{B} : \ell_1}\ \mathrm{id}}{-, \ell_1 \leqslant \ell_2, \mathrm{B} : \ell_1 \vdash \mathrm{B} : \ell_1}\ \mathrm{W_L}}{-, \ell_1 \leqslant \ell_2, \mathrm{A} : \ell_3, \mathrm{B} : \ell_1 \vdash \mathrm{B} : \ell_2}\ \mathrm{K_R}}{-, \mathfrak{m}(\mathrm{m}, \ell_1) \leqslant \ell_2, \mathrm{A} : \ell_3, \mathrm{B} : \ell_1 \vdash \mathrm{B} : \ell_2}\ \mathrm{P}^2_{\mathfrak{m}}
}{
\dfrac{\dfrac{\mathfrak{m}(\ell_3, \ell_4) \leqslant \ell_1, \mathfrak{m}(\mathrm{m}, \ell_1) \leqslant \ell_2, \mathrm{A} : \ell_3, \mathrm{A} \mathrel{-\!\!*} \mathrm{B} : \ell_4 \vdash \mathrm{B} : \ell_2}{\mathfrak{m}(\mathrm{m}, \ell_1) \leqslant \ell_2, \mathrm{A} * (\mathrm{A} \mathrel{-\!\!*} \mathrm{B}) : \ell_1 \vdash \mathrm{B} : \ell_2}\ {*_{\mathrm{L}}}}{\vdash (\mathrm{A} * (\mathrm{A} \mathrel{-\!\!*} \mathrm{B})) \mathrel{-\!\!*} \mathrm{B} : \mathrm{m}}\ {-\!\!*_{\mathrm{R}}}
}\ {-\!\!*_{\mathrm{L}}}
$$

From LBI-Proofs to GBI-Proofs

Translating Bunches and Sequents

Translating Bunches and Sequents

Given a bunch Γ and a letter δ, induction on the structure of Γ

Translating Bunches and Sequents

Given a bunch Γ and a letter δ, induction on the structure of Γ

- $\mathit{\Theta}(\mathrm{A}, \delta) = \{\, \mathrm{A} : \delta \,\}$
- $\mathit{\Theta}(\varnothing_{\mathfrak{a}}, \delta) = \{\, \mathrm{a} \leqslant \delta \,\}$ (or also $\emptyset$)
- $\mathit{\Theta}(\varnothing_{\mathfrak{m}}, \delta) = \{\, \mathrm{m} \leqslant \delta \,\}$

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- $\mathit{\Theta}((\Delta_1 \,; \Delta_2), \delta) = \mathit{\Theta}(\Delta_1, \delta 0) \cup \mathit{\Theta}(\Delta_2, \delta 1) \cup \{\, \mathfrak{a}(\delta 0, \delta 1) \leqslant \delta \,\}$
- $\mathit{\Theta}((\Delta_1 \,, \Delta_2), \delta) = \mathit{\Theta}(\Delta_1, \delta 0) \cup \mathit{\Theta}(\Delta_2, \delta 1) \cup \{\, \mathfrak{m}(\delta 0, \delta 1) \leqslant \delta \,\}$

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- $\Theta((\Delta_1 \,,\, \Delta_2), \delta) = \Theta(\Delta_1, \delta 0) \cup \Theta(\Delta_2, \delta 1) \cup \{\, \mathfrak{m}(\delta 0, \delta 1) \leqslant \delta \,\}$

Notation: $\Theta(\Gamma, \delta) = \Gamma : \delta$

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- $\Theta(\mathrm{A}, \delta) = \{ \mathrm{A} : \delta \}$
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- $\Theta((\Delta_1 ; \Delta_2), \delta) = \Theta(\Delta_1, \delta 0) \cup \Theta(\Delta_2, \delta 1) \cup \{ \mathfrak{a}(\delta 0, \delta 1) \leqslant \delta \}$
- $\Theta((\Delta_1 , \Delta_2), \delta) = \Theta(\Delta_1, \delta 0) \cup \Theta(\Delta_2, \delta 1) \cup \{ \mathfrak{m}(\delta 0, \delta 1) \leqslant \delta \}$

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Given a LBI sequent $\Gamma \vdash \mathrm{A}$

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Given a LBI sequent $\Gamma \vdash \mathrm{A}$

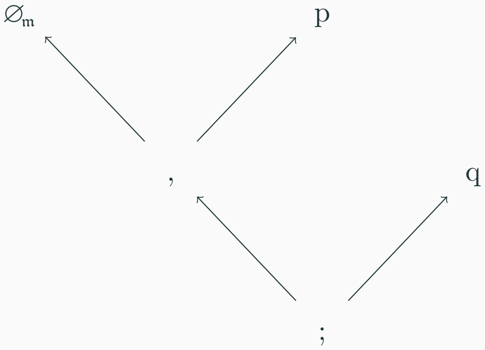
$$\mathit{\Theta}(\Gamma \vdash \mathrm{A}, \delta) \quad = \quad \mathit{\Theta}(\Gamma, \delta) \vdash \mathrm{A} : \delta \quad = \quad \Gamma : \delta \vdash \mathrm{A} : \delta$$

Translating Bunches and Sequents: an Example

$$(\varnothing_{\mathfrak{m}}\ ,\ \mathrm{p})\ ;\ \mathrm{q} \vdash \mathrm{r}$$

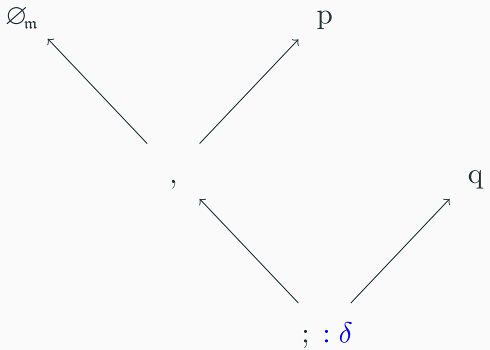
Translating Bunches and Sequents: an Example

$$(\varnothing_{\mathfrak{m}}\ ,\ \mathrm{p})\ ;\ \mathrm{q} \vdash \mathrm{r}$$



Translating Bunches and Sequents: an Example

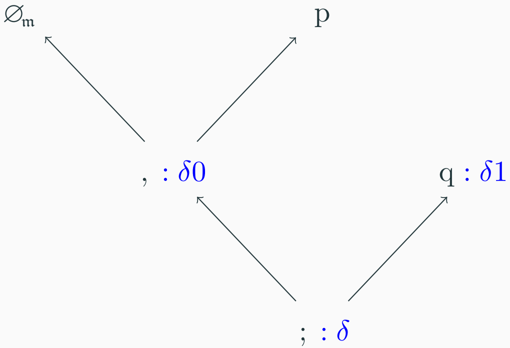
$$(\varnothing_{\mathfrak{m}}\ ,\ \mathrm{p})\ ;\ \mathrm{q} \vdash \mathrm{r}$$



$$\vdash \quad \mathrm{r} : \delta$$

Translating Bunches and Sequents: an Example

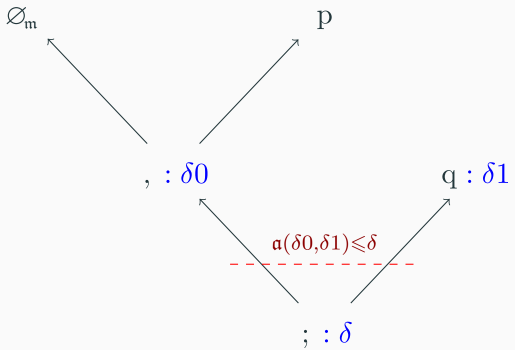
$$(\varnothing_{\mathfrak{m}}\ ,\ \mathrm{p})\ ;\ \mathrm{q} \vdash \mathrm{r}$$



$$\mathrm{q} : \delta 1 \quad \vdash \quad \mathrm{r} : \delta$$

Translating Bunches and Sequents: an Example

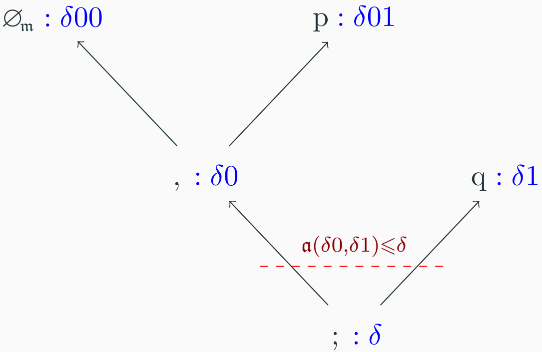
$$(\varnothing_{\mathfrak{m}}\ ,\ \mathrm{p})\ ;\ \mathrm{q} \vdash \mathrm{r}$$



$$\mathfrak{a}(\delta 0, \delta 1) \leqslant \delta, \qquad \mathrm{q} : \delta 1 \ \vdash \ \mathrm{r} : \delta$$

Translating Bunches and Sequents: an Example

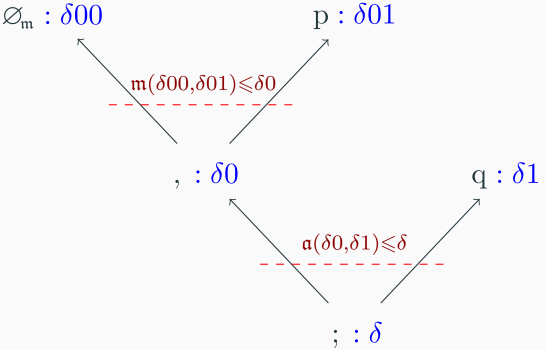
$$(\varnothing_{\mathfrak{m}}\ ,\ \mathrm{p})\ ;\ \mathrm{q} \vdash \mathrm{r}$$



$$\mathrm{m} \leqslant \delta 00, \qquad \mathfrak{a}(\delta 0, \delta 1) \leqslant \delta,\ \mathrm{p} : \delta 01,\ \mathrm{q} : \delta 1\ \vdash\ \mathrm{r} : \delta$$

Translating Bunches and Sequents: an Example

$$(\varnothing_{\mathfrak{m}}\ ,\ p)\ ;\ q \vdash r$$



$$m \leqslant \delta 00,\ \mathfrak{m}(\delta 00, \delta 01) \leqslant \delta 0,\ \mathfrak{a}(\delta 0, \delta 1) \leqslant \delta,\ p : \delta 01,\ q : \delta 1 \ \ \vdash \ \ r : \delta$$

From LBI-Proofs to GBI-Proofs

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The Translation Theorem

Each LBI proof can be translated to a GBI proof that follows the same rule application order.

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The Translation Theorem

Each LBI proof can be translated to a GBI proof that follows the same rule application order.

Proof.

- Induction on the height of the derivation
- Distinction on the last rule applied in LBI
- Given translations of the premises of a rule, there is a translation such that the translated conclusion is derivable in GBI from the translated premises.

□

From LBI to GBI: Case $*_{\mathrm{R}}$

$$\dfrac{\dfrac{\mathcal{D}_1}{\Gamma \vdash \mathrm{A}} \qquad \dfrac{\mathcal{D}_2}{\Delta \vdash \mathrm{B}}}{\Gamma , \Delta \vdash \mathrm{A} * \mathrm{B}} \; *_{\mathrm{R}} \text{ in LBI}$$

From LBI to GBI: Case $*_{\mathrm{R}}$

$$\dfrac{\dfrac{\mathcal{P}_1}{\Theta(\Gamma \vdash \mathrm{A}, \alpha)} \qquad \dfrac{\mathcal{P}_2}{\Theta(\Delta \vdash \mathrm{B}, \beta)}}{\Gamma, \Delta \vdash \mathrm{A} * \mathrm{B}} \; *_{\mathrm{R}} \text{ in LBI}$$

From LBI to GBI: Case $*_{\mathrm{R}}$

$$\dfrac{\dfrac{\mathcal{P}_1}{\Gamma : \alpha \vdash \mathrm{A} : \alpha} \qquad \dfrac{\mathcal{P}_2}{\Delta : \beta \vdash \mathrm{B} : \beta}}{\Gamma , \Delta \vdash \mathrm{A} * \mathrm{B}} \; *_{\mathrm{R}} \text{ in LBI}$$

From LBI to GBI: Case $*_{\mathrm{R}}$

$$\dfrac{\dfrac{\mathcal{P}_1}{\Gamma : \alpha \vdash \mathrm{A} : \alpha} \qquad \dfrac{\mathcal{P}_2}{\Delta : \beta \vdash \mathrm{B} : \beta}}{\Theta(\Gamma\,, \Delta \vdash \mathrm{A} * \mathrm{B}, \delta)}\ *_{\mathrm{R}} \text{ in LBI}$$

From LBI to GBI: Case $*_{\mathrm{R}}$

$$\dfrac{\dfrac{\mathcal{P}_1}{\Gamma : \alpha \vdash \mathrm{A} : \alpha} \qquad \dfrac{\mathcal{P}_2}{\Delta : \beta \vdash \mathrm{B} : \beta}}{\Gamma, \Delta : \delta \vdash \mathrm{A} * \mathrm{B} : \delta} \; *_{\mathrm{R}} \text{ in LBI}$$

From LBI to GBI: Case $*_{\mathrm{R}}$

$$\frac{\dfrac{\mathcal{P}_1}{\Gamma : \alpha \vdash \mathrm{A} : \alpha} \qquad \dfrac{\mathcal{P}_2}{\Delta : \beta \vdash \mathrm{B} : \beta}}{\mathfrak{m}(\delta 0, \delta 1) \leqslant \delta, \Gamma : \delta 0, \Delta : \delta 1 \vdash \mathrm{A} * \mathrm{B} : \delta}$$

From LBI to GBI: Case $*_{\mathrm{R}}$

$$\dfrac{\dfrac{\dfrac{\mathcal{P}_1}{\Gamma : \alpha \vdash \mathrm{A} : \alpha}}{\begin{array}{l}\mathfrak{m}(\delta 0, \delta 1) \leqslant \delta, \\ \Gamma : \delta 0, \Delta : \delta 1\end{array} \vdash \mathrm{A} : \delta 0} \qquad \dfrac{\dfrac{\mathcal{P}_2}{\Delta : \beta \vdash \mathrm{B} : \beta}}{\begin{array}{l}\mathfrak{m}(\delta 0, \delta 1) \leqslant \delta, \\ \Gamma : \delta 0, \Delta : \delta 1\end{array} \vdash \mathrm{B} : \delta 1}}{\mathfrak{m}(\delta 0, \delta 1) \leqslant \delta, \Gamma : \delta 0, \Delta : \delta 1 \vdash \mathrm{A} * \mathrm{B} : \delta} \; *_{\mathrm{R}} \text{ in GBI}$$

From LBI to GBI: Case $*_{\mathrm{R}}$

$$\dfrac{\dfrac{\dfrac{\mathcal{P}_1}{\Gamma : \alpha \vdash \mathrm{A} : \alpha}}{\Gamma : \delta 0 \vdash \mathrm{A} : \delta 0}\quad \dfrac{\dfrac{\mathcal{P}_2}{\Delta : \beta \vdash \mathrm{B} : \beta}}{\Delta : \delta 1 \vdash \mathrm{B} : \delta 1}}{}$$

$$\dfrac{\dfrac{\Gamma : \delta 0 \vdash \mathrm{A} : \delta 0}{\begin{matrix}\mathfrak{m}(\delta 0, \delta 1) \leqslant \delta, \\ \Gamma : \delta 0, \Delta : \delta 1\end{matrix} \vdash \mathrm{A} : \delta 0} \quad \dfrac{\Delta : \delta 1 \vdash \mathrm{B} : \delta 1}{\begin{matrix}\mathfrak{m}(\delta 0, \delta 1) \leqslant \delta, \\ \Gamma : \delta 0, \Delta : \delta 1\end{matrix} \vdash \mathrm{B} : \delta 1}}{\mathfrak{m}(\delta 0, \delta 1) \leqslant \delta, \Gamma : \delta 0, \Delta : \delta 1 \vdash \mathrm{A} * \mathrm{B} : \delta} *_{\mathrm{R}} \text{ in GBI}$$

From LBI to GBI: Case $*_{\mathrm{R}}$

$$
\dfrac{\mathcal{D}_1[\,\alpha \hookrightarrow \delta_0\,]}{\Gamma : \alpha \vdash \mathrm{A} : \alpha}
\qquad\qquad
\dfrac{\mathcal{D}_2[\,\beta \hookrightarrow \delta_1\,]}{\Delta : \beta \vdash \mathrm{B} : \beta}
$$

$$
\dfrac{
\dfrac{\Gamma : \delta 0 \vdash \mathrm{A} : \delta 0}{\begin{matrix}\mathfrak{m}(\delta 0, \delta 1) \leqslant \delta, \\ \Gamma : \delta 0, \Delta : \delta 1\end{matrix} \vdash \mathrm{A} : \delta 0}
\qquad
\dfrac{\Delta : \delta 1 \vdash \mathrm{B} : \delta 1}{\begin{matrix}\mathfrak{m}(\delta 0, \delta 1) \leqslant \delta, \\ \Gamma : \delta 0, \Delta : \delta 1\end{matrix} \vdash \mathrm{B} : \delta 1}
}{
\mathfrak{m}(\delta 0, \delta 1) \leqslant \delta, \Gamma : \delta 0, \Delta : \delta 1 \vdash \mathrm{A} * \mathrm{B} : \delta
}\; *_{\mathrm{R}} \text{ in GBI}
$$

From LBI to GBI: Case $*_{\mathrm{R}}$

$$\dfrac{\mathcal{D}_1[\,\alpha \hookrightarrow \delta_0\,]}{\dfrac{\overline{\Gamma : \delta 0 \vdash \mathrm{A} : \delta 0}}{\mathfrak{m}(\delta 0, \delta 1) \leqslant \delta, \Gamma : \delta 0, \Delta : \delta 1 \vdash \mathrm{A} : \delta 0}} \qquad \dfrac{\mathcal{D}_2[\,\beta \hookrightarrow \delta_1\,]}{\dfrac{\overline{\Delta : \delta 1 \vdash \mathrm{B} : \delta 1}}{\mathfrak{m}(\delta 0, \delta 1) \leqslant \delta, \Gamma : \delta 0, \Delta : \delta 1 \vdash \mathrm{B} : \delta 1}}$$

$$\dfrac{\mathfrak{m}(\delta 0, \delta 1) \leqslant \delta, \Gamma : \delta 0, \Delta : \delta 1 \vdash \mathrm{A} : \delta 0 \qquad \mathfrak{m}(\delta 0, \delta 1) \leqslant \delta, \Gamma : \delta 0, \Delta : \delta 1 \vdash \mathrm{B} : \delta 1}{\mathfrak{m}(\delta 0, \delta 1) \leqslant \delta, \Gamma : \delta 0, \Delta : \delta 1 \vdash \mathrm{A} * \mathrm{B} : \delta} \; *_{\mathrm{R}} \text{ in GBI}$$

From LBI to GBI: Case $\varnothing_{\mathfrak{m}}\uparrow$ (Shallow)

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Suppose we have a proof

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We then construct the following proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{P}[\,\alpha \hookrightarrow \delta 1\,]}{\Delta : \delta 1 \vdash \mathrm{A} : \delta 1}}{\delta 1 \leqslant \delta, -, \mathrm{m} \leqslant \delta 0, \Delta : \delta 1 \vdash \mathrm{A} : \delta 1}\ \mathrm{W_L}}{\delta 1 \leqslant \delta, -, \mathrm{m} \leqslant \delta 0, \Delta : \delta 1 \vdash \mathrm{A} : \delta}\ \mathrm{K_R}}{\mathfrak{m}(\mathrm{m}, \delta 1) \leqslant \delta, -, \mathrm{m} \leqslant \delta 0, \Delta : \delta 1 \vdash \mathrm{A} : \delta}\ \mathrm{P}^2_{\mathfrak{m}}}{\mathfrak{m}(\mathrm{m}, \delta 1) \leqslant \mathfrak{m}(\delta 0, \delta 1), \mathfrak{m}(\delta 0, \delta 1) \leqslant \delta, \mathrm{m} \leqslant \delta 0, \Delta : \delta 1 \vdash \mathrm{A} : \delta}\ \mathrm{T}}{\mathfrak{m}(\delta 0, \delta 1) \leqslant \delta, \mathrm{m} \leqslant \delta 0, \Delta : \delta 1 \vdash \mathrm{A} : \delta}\ \mathrm{C}^1_{\mathfrak{m}}$$

From GBI-Proofs to LBI-proofs

Translation Patterns

Translation Patterns

One to Many

$\bot_{\mathrm{L}}$ (Shallow)	$\leadsto$	$\bot_{\mathrm{L}}\ \bot_{\mathrm{R}}$
$\bot_{\mathrm{L}}$ (Deep)	$\leadsto$	$\bot_{\mathrm{L}}\ (\mathrm{C}^{\mathrm{i}}_{\mathfrak{t}}\ \ \mathrm{P}^{\mathrm{i}}_{\mathfrak{t}}\ )^{+}\ \bot_{\mathrm{R}}$
W (Shallow)	$\leadsto$	$\mathrm{P}^{1}_{\mathfrak{a}}\ \ \mathrm{K}_{\mathrm{R}}\ \sout{\mathrm{W}_{\mathrm{L}}}$
W (Deep)	$\leadsto$	$\mathrm{P}^{1}_{\mathfrak{a}}\ \ \mathrm{C}^{\mathrm{i}}_{\mathfrak{t}}\ \ \sout{\mathrm{W}_{\mathrm{L}}}$
C	$\leadsto$	$(\mathrm{C}_{\mathrm{L}}\ \mathrm{I}_{\mathfrak{a}})$ or C_{T}
$\mathrm{U}_{\mathfrak{r}}\!\uparrow$ (Shallow)	$\leadsto$	$\mathrm{C}^{1}_{\mathfrak{r}}\ \ \mathrm{P}^{2}_{\mathfrak{r}}\ \mathrm{K}_{\mathrm{R}}\ \sout{\mathrm{W}_{\mathrm{L}}}$
$\mathrm{U}_{\mathfrak{r}}\!\uparrow$ (Deep)	$\leadsto$	$\mathrm{C}^{1}_{\mathfrak{r}}\ \ \mathrm{P}^{2}_{\mathfrak{r}}\ \mathrm{C}^{\mathrm{i}}_{\mathfrak{t}}\ \ \sout{\mathrm{W}_{\mathrm{L}}}$
$\mathrm{U}_{\mathfrak{r}}\!\downarrow$	$\leadsto$	$(\mathrm{R}\ \mathrm{U}^{1}_{\mathfrak{r}})$ or $\mathrm{Z}^{1}_{\mathfrak{m}}$

Translation Patterns

One to Many

$\bot_{\mathrm{L}}$ (Shallow)	$\rightsquigarrow$	$\bot_{\mathrm{L}}\ \bot_{\mathrm{R}}$
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C	$\rightsquigarrow$	$(\mathrm{C}_{\mathrm{L}}\ \mathrm{I}_{\mathfrak{a}})$ or C_{T}
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$\mathrm{U}_{\mathfrak{r}}{\downarrow}$	$\rightsquigarrow$	$(\mathrm{R}\ \mathrm{U}^{1}_{\mathfrak{r}})$ or $\mathrm{Z}^{1}_{\mathfrak{m}}$

One to One

$\mathcal{R}$	$\rightsquigarrow$	$\mathcal{R}\ \sout{\mathrm{W}_{\mathrm{L}}}$	if $\mathcal{R} \in \{ \twoheadrightarrow\!\!*_{\mathrm{L}}, \rightarrow_{\mathrm{L}}, *_{\mathrm{R}}, \wedge_{\mathrm{R}} \}$
$\mathcal{R}$	$\rightsquigarrow$	$\mathcal{R}$	otherwise

Properties of LBI to GBI Translation

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- All logical rules are one to one
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- Label m has no predecessors but label a
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Restricted use of GBI rules

- "Safe" weakenings
- "Tree-like" contractions

$$\frac{\Gamma(\mathfrak{a}(\alpha s0, \delta s1) \leqslant \alpha s, \Theta : \alpha s0, \Theta : \alpha s1) : \alpha \vdash \Delta : \alpha}{\Gamma(\Theta : \alpha s) : \alpha \vdash \Delta : \alpha}\,\mathrm{C_T}$$

with $s \in \{0, 1\}^*$

Subterm and Reduction Relations

Let $\Gamma \vdash \mathrm{A}$ be sequent with label letters in a set L.

For $\mathfrak{r} \in \{\, \mathfrak{a}, \mathfrak{m} \,\}$

- Γ induces a **subterm relation** $\multimap = (\multimap_{\mathfrak{a}} \cup \multimap_{\mathfrak{m}})$

$$\ell_0 \multimap_{\mathfrak{r}} \ell_1 \text{ iff } \begin{cases} \ell_1 \in L \\ \exists\, \ell_2(\mathfrak{r}(\ell_1, \ell_2) \leqslant \ell_0 \in \Gamma \text{ or } \mathfrak{r}(\ell_2, \ell_1) \leqslant \ell_0 \in \Gamma) \end{cases}$$

- Γ induces a **reduction relation** $\rightsquigarrow$

$$\ell_0 \rightsquigarrow \ell_1 \text{ iff } \begin{cases} \ell_1 \leqslant \ell_0 \in \Gamma \\ \ell_0 \in L \\ \ell_1 \in L \cup \{\, \mathrm{m}, \mathrm{a}, \varpi \,\} \\ \ell_1 \neq \ell_0 \end{cases}$$

Reducibility and Normal Forms

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- A **reduction** of ℓ_0 to ℓ_n in Γ is

  a path $\ell_0 \rightsquigarrow \ell_1 \ldots \rightsquigarrow \ell_n$

  such that

  for all $0 \leqslant i < n$, $\ell_i \rightsquigarrow \ell_{i+1}$ in Γ

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- A reduction of ℓ_0 to ℓ_n is **minimal** if ℓ_n is irreducible

- If all minimal reductions of ℓ_0 terminate with the same irreducible label ℓ_n, then ℓ_n is the **normal form** of ℓ_0 (in Γ)

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  - ℓ_1 is reachable from ℓ_0 and
  - no formula A and no irreducible ℓ' on the path from ℓ_0 to ℓ_1 such that $\mathrm{A} : \ell' \in \Gamma$

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- for all $\ell_1 \leqslant \ell_0 \in \Gamma$,

  ℓ_0 is a label letter and if so is ℓ_1 then $\ell_0 \multimap \ell_1$
- for all $\mathfrak{r}(\ell_1, \ell_2) \leqslant \ell_0 \in \Gamma$, ℓ_1 and ℓ_2 are atomic
- if $\ell \succ \ell_0$ and ℓ_0 is reducible

  then ℓ_0 has a normal form and Γ has no ℓ_0-leaf
- if $\ell \succ \ell_0$ and ℓ_0 is irreducible, Γ has exactly one ℓ_0-leaf
- the set $\{ \ell_1 \multimap \ell_0 \mid \ell \succ \ell_0 \}$ is a **tree with root** ℓ in which all internal nodes have exactly two children of the same $\mathfrak{r}$ type.

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A GBI-proof has the **tree property** iff all of its sequents have it.

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Given a finite set B of bunches

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- $\mathcal{B}_{\mathfrak{a}}(B) = \varnothing_{\mathfrak{a}}$ if B is empty
- $\mathcal{B}_{\mathfrak{a}}(B) = B_1 \,;\, \ldots \,;\, B_n$ with $B_i \in B$ otherwise

Similarly for $\mathcal{B}_{\mathfrak{m}}(B)$ w.r.t. $\varnothing_{\mathfrak{m}}$ and " , "

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Given $\Gamma \vdash \mathrm{A} : \ell$ in a normal GBI-proof, $\mathfrak{B}(\Gamma \vdash \mathrm{A} : \ell) = \Gamma @ \ell \vdash \mathrm{A}$

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Given $\Gamma \vdash \mathrm{A} : \ell$ in a normal GBI-proof, $\mathfrak{B}(\Gamma \vdash \mathrm{A} : \ell) = \Gamma @ \ell \vdash \mathrm{A}$

- $\Gamma @ \mathrm{m} = \varnothing_{\mathrm{m}}$, $\Gamma @ \mathrm{a} = \varnothing_{\mathrm{a}}$, $\Gamma @ \varpi = \bot$
- $\Gamma @ \ell = \Gamma @ \ell'$ if for some ℓ', $\ell \rightsquigarrow \ell'$ in Γ
- let $\mathrm{L} = \{\, \mathrm{A}_i \mid \mathrm{A}_i : \ell \in \Gamma \,\}$ and $\mathrm{S}_{\mathfrak{r}} = \{\, \ell_i \mid \ell \multimap_{\mathfrak{r}} \ell_i \text{ in } \Gamma \,\}$

$$\Gamma @ \ell = \begin{cases} \mathcal{B}_{\mathfrak{a}}(\mathrm{L}) & \text{if } \ \mathrm{L} \neq \varnothing \\ \mathcal{B}_{\mathfrak{a}}(\mathrm{S}_{\mathfrak{a}}) & \text{if } \ \mathrm{L} = \varnothing, \mathrm{S}_{\mathfrak{m}} = \varnothing, \mathrm{S}_{\mathfrak{a}} \neq \varnothing \\ \mathcal{B}_{\mathfrak{m}}(\mathrm{S}_{\mathfrak{m}}) & \text{if } \ \mathrm{L} = \varnothing, \mathrm{S}_{\mathfrak{m}} \neq \varnothing, \mathrm{S}_{\mathfrak{a}} = \varnothing \end{cases}$$

Translating GBI-proofs

Let $\mathcal{D}$ be a GBI-derivation with the tree property.

- Translate all GBI inference rule instances

$$\frac{\mathrm{P_1} \quad \ldots \quad \mathrm{P_n}}{\mathrm{C}}\ \mathrm{R}$$

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into LBI inference rule instances

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  where $\mathfrak{B}(\mathrm{R})$ is "$\equiv$" when R has no LBI counterpart

- Clean up (prune) the proof to remove stuttering

Translation Algorithm: an Example

Apply translation algorithm top-down

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{P}}{\Delta : \delta 1 \vdash \mathrm{A} : \delta 1}}{\delta 1 \leqslant \delta, -, \mathrm{m} \leqslant \delta 0, \Delta : \delta 1 \vdash \mathrm{A} : \delta 1}\ \mathrm{W_L}}{\delta 1 \leqslant \delta, -, \mathrm{m} \leqslant \delta 0, \Delta : \delta 1 \vdash \mathrm{A} : \delta}\ \mathrm{K_R}}{\mathfrak{m}(\mathrm{m}, \delta 1) \leqslant \delta, -, \mathrm{m} \leqslant \delta 0, \Delta : \delta 1 \vdash \mathrm{A} : \delta}\ \mathrm{P}^2_{\mathfrak{m}}}{\mathfrak{m}(\mathrm{m}, \delta 1) \leqslant \delta, \mathfrak{m}(\delta 0, \delta 1) \leqslant \delta, \mathrm{m} \leqslant \delta 0, \Delta : \delta 1 \vdash \mathrm{A} : \delta}\ \mathrm{T}
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$$

$$
\dfrac{\mathfrak{m}(\mathrm{m},\delta 1) \leqslant \delta, \mathfrak{m}(\delta 0,\delta 1) \leqslant \delta, \mathrm{m} \leqslant \delta 0, \Delta : \delta 1 \vdash \mathrm{A} : \delta}{\mathfrak{m}(\delta 0,\delta 1) \leqslant \delta, \mathrm{m} \leqslant \delta 0, \Delta : \delta 1 \vdash \mathrm{A} : \delta}\,\mathrm{C}^{1}_{\mathfrak{m}}
$$

Translation Algorithm: an Example

Apply translation algorithm top-down

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}}{\Delta \vdash \mathrm{A}}}{\Delta \vdash \mathrm{A}}\,\mathrm{W}}{\Delta \vdash \mathrm{A}}\,\equiv}{\varnothing_{\mathfrak{m}}\,, \Delta \vdash \mathrm{A}}\,\mathrm{U}_{\mathfrak{m}}\!\uparrow}{\varnothing_{\mathfrak{m}}\,, \Delta \vdash \mathrm{A}}\,\equiv}{\mathfrak{m}(\delta 0, \delta 1) \leqslant \delta, \mathrm{m} \leqslant \delta 0, \Delta : \delta 1 \vdash \mathrm{A} : \delta}\,\mathrm{C}^{1}_{\mathfrak{m}}$$

Translation Algorithm: an Example

Apply translation algorithm top-down

$$
\begin{array}{cl}
\mathcal{D} & \\
\hline
\Delta \vdash \mathrm{A} & \\
\hline
\Delta \vdash \mathrm{A} & \mathrm{W} \\
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\Delta \vdash \mathrm{A} & \equiv \\
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\varnothing_{\mathfrak{m}}\,, \Delta \vdash \mathrm{A} & \mathrm{U}_{\mathfrak{m}}\!\uparrow \\
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\varnothing_{\mathfrak{m}}\,, \Delta \vdash \mathrm{A} & \equiv \\
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\varnothing_{\mathfrak{m}}\,, \Delta \vdash \mathrm{A} & \equiv
\end{array}
$$

Translation Algorithm: an Example

Remove stuttering

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}}{\Delta \vdash \mathrm{A}}}{\Delta \vdash \mathrm{A}}\,\mathrm{W}}{\varnothing_{\mathfrak{m}}\,,\Delta \vdash \mathrm{A}}\,\equiv}{\varnothing_{\mathfrak{m}}\,,\Delta \vdash \mathrm{A}}\,\equiv}{\varnothing_{\mathfrak{m}}\,,\Delta \vdash \mathrm{A}}\,\equiv}{\varnothing_{\mathfrak{m}}\,,\Delta \vdash \mathrm{A}}\,\equiv$$

Translation Algorithm: an Example

Remove stuttering

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}}{\Delta \vdash \mathrm{A}}}{\Delta \vdash \mathrm{A}}\,\mathrm{W}}{\varnothing_{\mathfrak{m}}\,, \Delta \vdash \mathrm{A}} \equiv}{\varnothing_{\mathfrak{m}}\,, \Delta \vdash \mathrm{A}} \equiv}{\varnothing_{\mathfrak{m}}\,, \Delta \vdash \mathrm{A}} \equiv$$

Translation Algorithm: an Example

Remove stuttering

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}}{\Delta \vdash \mathrm{A}}}{\Delta \vdash \mathrm{A}}\,\mathrm{W}}{\varnothing_{\mathfrak{m}}\,,\Delta \vdash \mathrm{A}}\equiv}{\varnothing_{\mathfrak{m}}\,,\Delta \vdash \mathrm{A}}\equiv$$

Translation Algorithm: an Example

Remove stuttering

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{\Delta \vdash \mathrm{A}}}{\Delta \vdash \mathrm{A}}\,\mathrm{W}}{\varnothing_{\mathfrak{m}}\,, \Delta \vdash \mathrm{A}} \equiv$$

Translation Algorithm: an Example

Remove stuttering

$$\cfrac{\cfrac{\mathcal{D}}{\Delta \vdash \mathrm{A}}}{\varnothing_{\mathfrak{m}}\,, \Delta \vdash \mathrm{A}} \equiv$$

Conclusion and Future Work

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- An alternative semantics for BI
- GBI labelled sequent calculus
- Translation from LBI to GBI
- Translation from GBI with tree property to LBI

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Future Work

- Normalization of multi-conclusioned GBI
- Effective procedure translating TBI-proofs into GBI-proofs
- Completeness of KRM wrt LBI