Bunched hypersequent calculi for distributive substructural logics

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Bunched (hyper)sequent calculi for distributive substructural logics

> *How can we generate cutfree proof calculi with the subformula property for axiomatic extensions of DFL_e?*

- Underlying aim: proof calculi as a tool for proving results about the logics.
- We proposed an answer by extending the bunched calculi with a “hyper" structure (LPAR 2017)
- The ensuing proof calculi extend the base calculus for DFL_e by analytic structural rules, and cut-elimination was proved
- modularity: various combinations of axiomatic extensions $\leftrightsquigarrow$ corresponding combination of analytic structural rules

Throughout, for brevity I write “structural rule" for “analytic structural rule"

- We then extended the result to formulate logics extending Bunched Implication (BI) and in the vicinity of Boolean Bunched Implication (BBI)
- We present this material here recognising the expertise and interest in bunched logics within the Ticamore community

Substructural logics. The Lambek calculus with exchange FL_e

- Remove some of the properties of the structural connective comma from the intuitionistic calculus LJ to obtain substructural logics
- FL_e: delete contraction and weakening rules. Differentiates multiplicative and additive connectives $\otimes$ and $\wedge$ (that conflate in presence of (w) and (c)).

$$p \Rightarrow p \qquad \bot, \Gamma \Rightarrow D \qquad \varnothing_m \Rightarrow 1 \qquad \Gamma \Rightarrow \top \qquad \frac{A_1, A_2, \Gamma \Rightarrow D}{A_1 \otimes A_2, \Gamma \Rightarrow D}\otimes\mathrm{l}$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B}\otimes\mathrm{r} \qquad \frac{\Gamma \Rightarrow A \qquad B, \Delta \Rightarrow D}{A \multimap B, \Gamma, \Delta \Rightarrow D}\multimap\mathrm{l} \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \multimap B}\multimap r$$

$$\frac{A_i, \Gamma \Rightarrow D}{A_1 \wedge A_2, \Gamma \Rightarrow D}\wedge l \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\wedge r \qquad \frac{\Gamma, A, B, \Delta \Rightarrow D}{\Delta, B, A, \Gamma \Rightarrow D}\mathrm{e}$$

$$\frac{A, \Gamma \Rightarrow D \qquad B, \Gamma \Rightarrow D}{A \vee B, \Gamma \Rightarrow D}\vee l \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\vee r \qquad \genfrac{}{}{1.5pt}{}{\Gamma \Rightarrow A}{\Gamma, \varnothing_m \Rightarrow A}\varnothing_m\mathrm{I/E}$$

- antecedent: comma-separated list of formulas
- Cut-elimination is well-known so we omitted the cut rule (throughout, where present, the cut rule will be explicitly stated)

An observation: $\mathbf{FL}_e$ is not distributive

$$A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C) \text{ is not derivable}$$

Proof: exhaustive backward proof search (none of the following works)

$$\frac{A \Rightarrow (A \wedge B) \vee (A \wedge C)}{A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)} \wedge\mathrm{l}$$

$$\frac{B \vee C \Rightarrow (A \wedge B) \vee (A \wedge C)}{A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)} \wedge\mathrm{l}$$

$$\frac{A \wedge (B \vee C) \Rightarrow A \wedge B}{A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)} \vee\mathrm{r}$$

$$\frac{A \wedge (B \vee C) \Rightarrow A \wedge C}{A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)} \vee\mathrm{r}$$

- The need for distributivity arises e.g. in relevant logics.
- There is no structural rule that can be added to FL_e to obtain the distributivity axiom (Ciabattoni, Galatos, Terui 2012)

A bunched calculus $s\mathrm{DFL}_e$ for DFL_e (Dunn 1974, Mints 1976)

- antecedent is given by following grammar (bunch):

$$\mathfrak{B} := \text{formula} \mid \varnothing_a \mid \varnothing_m \mid (\mathfrak{B}, \mathfrak{B}) \mid (\mathfrak{B}; \mathfrak{B}) \qquad \Gamma[\Delta] \text{ denotes that } \Delta \text{ is a sub-bunch}$$

$\otimes$, $-\!\!*$, 1, $\varnothing_m$, comma (multiplicative) $\qquad$ $\vee$, $\wedge$, $\top$, $\bot$, $\varnothing_a$, semicolon (additive)

$$p \Rightarrow p \qquad \bot, \Gamma \Rightarrow D \qquad \varnothing_m \Rightarrow 1 \qquad \varnothing_a \Rightarrow \top \qquad \frac{\Gamma[A_1, A_2] \Rightarrow D}{\Gamma[A_1 \otimes A_2] \Rightarrow D}\otimes l$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B}\otimes r \quad \frac{\Gamma \Rightarrow A \qquad \Sigma[B] \Rightarrow D}{\Sigma[\Gamma, A -\!\!* B] \Rightarrow D} -\!\!* l \quad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A -\!\!* B} -\!\!* r$$

$$\frac{\Gamma[A_1; A_2] \Rightarrow D}{\Gamma[A_1 \wedge A_2] \Rightarrow D}\wedge l \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\wedge r \qquad \frac{\Sigma[\Gamma, \Delta] \Rightarrow A}{\Sigma[\Delta, \Gamma] \Rightarrow A}\text{(m-e)}$$

$$\frac{\Gamma[A] \Rightarrow D \qquad \Gamma[B] \Rightarrow D}{\Gamma[A \vee B] \Rightarrow D}\vee l \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\vee r \qquad \frac{\Sigma[(X, Y), Z] \Rightarrow A}{\Sigma[X, (Y, Z)] \Rightarrow A}\text{(m-as)}$$

$$\frac{\Sigma[(X; Y); Z] \Rightarrow A}{\Sigma[X; (Y; Z)] \Rightarrow A}\text{(a-as)} \qquad \frac{\Sigma[X; Y] \Rightarrow A}{\Sigma[Y; X] \Rightarrow A}\text{(a-ex)} \qquad \frac{\Sigma[X] \Rightarrow A}{\Sigma[X; Y] \Rightarrow A}\text{(a-w)}$$

$$\frac{\Sigma[X; X] \Rightarrow A}{\Sigma[X] \Rightarrow A}\text{(a-ctr)} \qquad \frac{\Sigma[X] \Rightarrow A}{\Sigma[X, \varnothing_m] \Rightarrow A}\varnothing_m\text{I/E} \qquad \frac{\Sigma[X] \Rightarrow A}{\Sigma[X; \varnothing_a] \Rightarrow A}\varnothing_a\text{I/E}$$

Distributivity $A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$ **is derivable in** $s\mathbf{DFL}_e$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A; B \Rightarrow A} \qquad \dfrac{B \Rightarrow B}{A; B \Rightarrow B}}{A; B \Rightarrow A \wedge B}}{A; B \Rightarrow (A \wedge B) \vee (A \wedge C)} \qquad \dfrac{\dfrac{\dfrac{A \Rightarrow A}{A; C \Rightarrow A} \qquad \dfrac{C \Rightarrow C}{A; C \Rightarrow C}}{A; C \Rightarrow A \wedge C}}{A; C \Rightarrow (A \wedge B) \vee (A \wedge C)}}{A; B \vee C \Rightarrow (A \wedge B) \vee (A \wedge C)}}{A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)}
$$

General method generating (hyper)sequent structural rules from axioms

- (Ciabattoni, Galatos, Terui 2008) developed a general method for generating cutfree (hyper)sequent calculi by computing structural rules from the axioms

We adapt this method to the bunched (hyper)sequent calculi:

- Specifically this entails
  (i) Interpret the additional structure and prove a cut-elimination theorem on this extended structure.
  (ii) (extending the CGT2008 algorithm for transforming an axiom into a structural rule)
  (iii) Characterise those axiom extensions that can be presented
  (iv) Later, we adapt to bunched implication logics (DFL_e with two implications defined on $\Rightarrow$) where the above interpretation does not hold.

Example: A calculus for $\mathbf{DFL}_e + (1 \wedge (p \otimes q)) \multimap p$

- $(1 \wedge (p \otimes q)) \multimap p$ is the restricted weakening axiom

STEP 1: use invertible rules backwards:

$$\dfrac{\dfrac{\dfrac{\varnothing_m, (\varnothing_m; (p, q)) \Rightarrow p}{1, (1; (p, q)) \Rightarrow p}}{1, (1 \wedge (p \otimes q)) \Rightarrow p}}{1 \Rightarrow (1 \wedge (p \otimes q)) \multimap p}$$

STEP 2: So it suffices to derive $\varnothing_m, (\varnothing_m; (p, q)) \Rightarrow p$. In the presence of cut the following equivalences hold ('Ackermann's lemma')

$$\varnothing_m, (\varnothing_m; (p, q)) \Rightarrow p \qquad\qquad \dfrac{X \Rightarrow p}{\varnothing_m, (\varnothing_m; (X, q)) \Rightarrow p}$$

$$\dfrac{X \Rightarrow p \qquad Y \Rightarrow q}{\varnothing_m, (\varnothing_m; (X, Y)) \Rightarrow p} \qquad\qquad \dfrac{X \Rightarrow p \quad Y \Rightarrow q \quad \Gamma[p] \Rightarrow B}{\varnothing_m, (\varnothing_m; (X, Y)) \Rightarrow B}$$

Example: A calculus for $\mathbf{DFL}_e + (1 \wedge (p \otimes q)) \multimap p$ (II)

- After Step 2 we obtained the following rule whose addition to $s\mathrm{DFL}_e + cut$ is equivalent to $\mathrm{DFL}_e + (1 \wedge (p \otimes q)) \multimap p$

$$\frac{X \Rightarrow p \quad Y \Rightarrow q \quad \Gamma[p] \Rightarrow B}{\varnothing_m, (\varnothing_m; (X, Y)) \Rightarrow B}$$

STEP 3: Apply all possible cuts to the premises (assuming termination) to get the equivalent rules

$$\frac{\Gamma[X] \Rightarrow B \quad Y \Rightarrow q}{\varnothing_m, (\varnothing_m; (X, Y)) \Rightarrow B} \qquad \frac{\Gamma[X] \Rightarrow B}{\varnothing_m, (\varnothing_m; (X, Y)) \Rightarrow B}\ \mathrm{r}$$

- $s\mathrm{DFL}_e + r + cut$ is sound and complete for $\mathrm{DFL}_e + (1 \wedge (p \otimes q)) \multimap p$.
- Prove cut-elimination theorem for analytic structural rule ext.: $s\mathrm{DFL}_e + r$.
- $s\mathrm{DFL}_e + r$ is a proof calculus for $\mathrm{DFL}_e + (1 \wedge (p \otimes q)) \multimap p$ with the subformula property.

An example where the argument fails

$$\mathrm{DFL}_e + (p \multimap 0) \vee ((p \multimap 0) \multimap 0)$$

- Applying invertible rules to $1 \Rightarrow (p \multimap 0) \vee ((p \multimap 0) \multimap 0)$ we get

$$\varnothing_m \Rightarrow (p \multimap 0) \vee ((p \multimap 0) \multimap 0)$$

- Applying Ackermann lemma (below left), then invertible rule ($\vee$l):

$$\frac{(p \multimap 0) \vee ((p \multimap 0) \multimap 0) \Rightarrow X}{\varnothing_m \Rightarrow X} \qquad \frac{(p \multimap 0) \Rightarrow X \qquad ((p \multimap 0) \multimap 0) \Rightarrow X}{\varnothing_m \Rightarrow X}$$

- The rule above right is not yet ready for the cutting step. . . the formulas in it need to be decomposed but no invertible rules apply.
- Structural rules extensions of $s\mathrm{DFL}_e$ are not expressive enough to present $\mathrm{DFL}_e + (p \multimap 0) \vee ((p \multimap 0) \multimap 0)$
- We need to extend the sequent formalism further. . .

Bunched hypersequent calculus for $\mathbf{DFL}_e + (p{-\!\!*}0) \vee ((p{-\!\!*}0){-\!\!*}0)$

- A natural extension of a sequent $\Gamma \Rightarrow A$ is to a non-empty set of sequents (Avron 1996, Pottinger 1983)

$$\Gamma_1 \Rightarrow A_1 \,|\, \Gamma_2 \Rightarrow A_2 \,|\, \ldots \,|\, \Gamma_{n+1} \Rightarrow A_{n+1}$$

- Here we take the analogous extension of $s\mathrm{DFL}_e$ with hypersquent structure
- The hypersequent calculus $h\mathrm{DFL}_e$ is obtained from $s\mathrm{DFL}_e$ as follows:

Add a hypersequent context "$g\,|$" to each rule. Also add rules manipulating the components

$$\frac{g\,|\,\Gamma, A \Rightarrow B}{g\,|\,\Gamma \Rightarrow A{-\!\!*}B}\ {-\!\!*}\mathrm{r} \qquad \frac{h\,|\,h\,|\,g}{h\,|\,g}\ \mathrm{EC} \qquad \frac{g}{h\,|\,g}\ \mathrm{EC}$$

Bunched hypersequent calculus for $\mathbf{DFL}_e + (p \multimap 0) \vee ((p \multimap 0) \multimap 0)$ (II)

- Prove soundness of $h\mathrm{DFL}_e$ wrt DFL_e interpreting | as disjunction
- Under this interpretation, we begin with the following hypersequent calculus

$$h\mathrm{DFL}_e + cut + g \,|\, 1 \Rightarrow p \multimap 0 \,|\, 1 \Rightarrow (p \multimap 0) \multimap 0$$

- Let us convert $g \,|\, 1 \Rightarrow p \multimap 0 \,|\, 1 \Rightarrow (p \multimap 0) \multimap 0$ into a structural rule...
- STEP 1: apply invertible rules:

$$g \,|\, 1 \Rightarrow p \multimap 0 \,|\, 1 \Rightarrow (p \multimap 0) \multimap 0 \qquad g \,|\, \varnothing_m, p \Rightarrow \mathit{O}_m \,|\, \varnothing_m, p \multimap 0 \Rightarrow \mathit{O}_m$$

Bunched hypersequent calculus for $\mathbf{DFL}_e + (p{-\!\!*}0) \vee ((p{-\!\!*}0){-\!\!*}0)$ (III)

- STEP 2: apply Ackermann's lemma to $g \,|\, \varnothing_m, p \Rightarrow \mathcal{O}_m \,|\, \varnothing_m, p{-\!\!*}0 \Rightarrow \mathcal{O}_m$:

$$\frac{g \,|\, X \Rightarrow p \qquad g \,|\, Y \Rightarrow p{-\!\!*}0}{g \,|\, \varnothing_m, X \Rightarrow \mathcal{O}_m \,|\, \varnothing_m, Y \Rightarrow \mathcal{O}_m}$$

- STEP 3: invertible rules and all possible cuts to obtain a structural rule

$$\frac{g \,|\, X \Rightarrow p \qquad g \,|\, p, Y \Rightarrow \mathcal{O}_m}{g \,|\, \varnothing_m, X \Rightarrow \mathcal{O}_m \,|\, \varnothing_m, Y \Rightarrow \mathcal{O}_m} \qquad \frac{g \,|\, X, Y \Rightarrow \mathcal{O}_m}{g \,|\, \varnothing_m, X \Rightarrow \mathcal{O}_m \,|\, \varnothing_m, Y \Rightarrow \mathcal{O}_m}\,\mathrm{r}$$

- $h\mathrm{DFL}_e + cut + r$ is a calculus for $\mathrm{DFL}_e + (p{-\!\!*}0) \vee ((p{-\!\!*}0){-\!\!*}0)$
- Cut-elimination can be proved for $h\mathrm{DFL}_e$ analytic structural rule ext.
- Thus $h\mathrm{DFL}_e + r$ is a calculus for $\mathrm{DFL}_e + (p{-\!\!*}0) \vee ((p{-\!\!*}0){-\!\!*}0)$

The substructural hierarchy over $\mathbf{DFL}_e$

- We can adapt the substructural hierarchy of (Ciabattoni, Galatos, Terui 2008) to extensions of DFL_e.
- Set $\mathcal{N}_0^d$ and $\mathcal{P}_0^d$ as the set of propositional variables. Define:

$$\begin{aligned}
\mathcal{P}_{n+1}^d &::= 1 \mid \mathcal{N}_n^d \mid \mathcal{P}_{n+1}^d \otimes \mathcal{P}_{n+1}^d \mid \mathcal{P}_{n+1}^d \wedge \mathcal{P}_{n+1}^d \mid \mathcal{P}_{n+1}^d \vee \mathcal{P}_{n+1}^d \\
\mathcal{N}_{n+1}^d &::= \mathcal{O}_m \mid \mathcal{P}_n^d \mid \mathcal{N}_{n+1}^d \wedge \mathcal{N}_{n+1}^d \mid \mathcal{P}_{n+1}^d \multimap \mathcal{N}_{n+1}^d
\end{aligned}$$

- The positive classes $\mathcal{P}_i$ contain formulae whose most external connective is invertible on the left
- The negative classes $\mathcal{N}_i$) contain formulae whose most external connective is invertible on the right

Theorem

Every extension of DFL_e by a disjunction of $\mathcal{N}_2^d$ axioms has a cutfree structural rule extension of $hDFL_e$ (provided the cuts on the premises terminate).

- $\mathcal{N}_2^d$: formulas whose non-invertible connectives are at the surface
- disjunctions of $\mathcal{N}_2^d$: disjunctions of formulas whose non-invertible…

The logic of bunched implications BI (O'Hearn and Pym, 1999)

- BI can be used for resource composition and systems modelling and as a propositional fragment of separation logic
- A proof calculus is obtained by extending $s\mathrm{DFL}_e$ with an intuitionistic implication $\to$
- This is the usual bunched calculus sBI for BI:

$$\frac{\Gamma \Rightarrow A \qquad \Sigma[B] \Rightarrow D}{\Sigma[\Gamma; A \to B] \Rightarrow D} \to\mathrm{l} \qquad \frac{A; \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B} \to r$$

$$\frac{\Gamma \Rightarrow A \qquad \Sigma[B] \Rightarrow D}{\Sigma[\Gamma, A {-\!\!*} B] \Rightarrow D} {-\!\!*}\mathrm{l} \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A {-\!\!*} B} {-\!\!*} r$$

- Algebraic semantics: generalised bunched implication GBI algebras (Galatos and Jipsen 2017).
- GBI algebra: Heyting algebra $((A, \leq), \vee, \wedge, \to, \bot, \top)$ extended with a commutative monoid $(A, \otimes, 1)$ and its residuated implication ${-\!\!*}$ (defined wrt $\leq$):

$$\text{i.e.} \quad x \otimes y \leq z \text{ iff } x \leq y {-\!\!*} z$$

A cutfree calculus for $s\mathrm{BI} + cut + \top \Rightarrow p \vee (p \to \bot)$ (BBI): an attempt

- Boolean BI (BBI) is the counterpart of BI with intuitionistic logic replaced by classical logic
- BBI is the propositional basis of separation logic (more widely used than BI)
- BBI is undecidable (Larchey-Wendling and Galmiche, 2010)
- We cannot get a sequent structural rule from $\top \Rightarrow p \vee (p \to \bot)$:
- The issue is that the algorithm on sequents can only handle non-invertible connectives at the surface…
- …and the non-invertible $\to$ is not at the surface (it is nested below $\vee$)
- Idea: add hypersequent structure to *s*BI

A cutfree calculus for $s\mathrm{BI} + cut + \top \Rightarrow p \vee (p \rightarrow \bot)$: an attempt (II)

- Introduce hypersequent structure by reading $\top \Rightarrow p \vee (p \rightarrow \bot)$ as $\top \Rightarrow p \mid \top \Rightarrow (p \rightarrow \bot)$
- Our cut-elimination proof extends to analytic structural rule ext. of $h\mathrm{BI}$

Theorem

Every analytic structural rule extension of hBI has cut-elimination.

- However: the two right implication rules do not permit a (formula) interpretation of $\Rightarrow$ so we cannot interpret the hypersequent...
- So we have completeness of the hypersequent calculus:

$$\{\Gamma \Rightarrow A \quad | \quad \Gamma \Rightarrow A \text{ derivable in } s\mathrm{BI} + cut + \mathcal{P}_2^{\mathrm{BI}} \Rightarrow \mathcal{N}_2^{\mathrm{BI}}\} \subseteq$$
$$\{\Gamma \Rightarrow A \quad | \quad \Gamma \Rightarrow A \text{ derivable in } h\mathrm{BI} + r\}$$

- but not the reverse inclusion

A new perspective: start from the rules

- For every structural rule r, the following set is well-defined

$$(h\mathrm{BI} + r)_{\mathrm{seq}} := \{\Gamma \Rightarrow A \quad | \quad \Gamma \Rightarrow A \text{ derivable in } h\mathrm{BI} + r\}$$

- Moreover, this set is closed under the cut-rule.

What can we say about this logic (consequence relation $\Rightarrow$)?

Future work, future collaborations?

- Specifically: add structural rule which derives desired sequent, and use the subformula property to check the consistency of structural rule extensions
- E.g. hBI $+$ (cl) below has cut-elimination and derives $1 \Rightarrow p \vee (p \rightarrow \bot)$

$$\frac{g \mid \Gamma[\Sigma] \Rightarrow \psi}{g \mid \Gamma[\varnothing_m] \Rightarrow \psi \mid \varnothing_m ; \Sigma \Rightarrow \mathcal{O}_a} \text{(cl)}$$

- consistency: exploiting absence of cut, observe that $\top \Rightarrow \bot$ is not derivable
- (backward proof search: there is no way of obtaining the semicolon-separated $\varnothing_m$ that is required for an application of (cl)).
- By a similar argument $\top \Rightarrow p \vee (p \rightarrow \bot)$ is not derivable...
- ...so $(h\mathrm{BI} + (cl))_{\mathrm{seq}}$ is not the logic BBI
- We can formulate other logics extending BI and in the vicinity of BBI by using other structural rules...

Future work, future collaborations? (II)

- Can we extend the semantics of BI to such logics?
- Can we find interesting resource interpretations for such logics?
- In this regard, an interesting option might be to replace intuitionistic logic in BI with an intermediate logic
- E.g. $h\mathrm{BI} + (com)$ derives the linearity axiom $\top \Rightarrow (p \to q) \vee (q \to p)$

$$\frac{g \mid \Gamma[Y] \Rightarrow \psi \qquad g \mid \Sigma[X] \Rightarrow \phi}{g \mid \Gamma[X] \Rightarrow \psi \mid \Sigma[Y] \Rightarrow \phi} \text{(com)}$$

- In this way, could we obtain decidable BBI-like logics?